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Articles by Title

Articles by Authors

Problems and Questions

Contributors

Numbers refer to pages. **Bold** type indicates a problem solved and solution published; *italics*, a problem solved but solution not published; ordinary type, a problem proposed.

Solutions

The page on which a solution appears is in the parentheses following the number of the problem.

Quickies

Trickies

Vol. 29, No. 1, Sept.-Oct., 1955

MATHEMATICS magazine

MATHEMATICS MAGAZINE

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MATHEMATICS MAGAZINE

VOL. 29, No. 1, Sept.-Oct. 1955

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("*Our Contributors*" will be brought up to date in the Nov.-Dec. issue

Mathematical Analyst *Keith Kersery loads jet transport flutter problem into one of Lockheed's two 701's. On order: two 704's to help keep Lockheed in forefront of numerical analysis and production control data processing.*

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COMBINATORIAL TOPOLOGY OF SURFACES*

Robert C. James

The first section of this paper will be devoted to the study of two particular surfaces. The methods used to treat these surfaces will then be extended to develop a classification of surfaces, each class consisting of combinatorially equivalent surfaces. Applications will be made to the study of the topological nature of covering surfaces and of Riemann surfaces in particular. It will be assumed that the reader is familiar with the material of Chapter V, pages 235-244 and 256-264, in the book *What Is Mathematics?* by Courant and Robbins. Other than for references to this book, no previous knowledge of Topology will be assumed.

1. Two Examples.

Example 1.1. Consider the surface of Figure 1. It can be described as a sphere with three appendages, a "handle", a "crosscap", and a "cuff". Cut the surface along the path which consists of path x followed by paths y and z in the directions indicated in Figure 2. After this cut has been made,

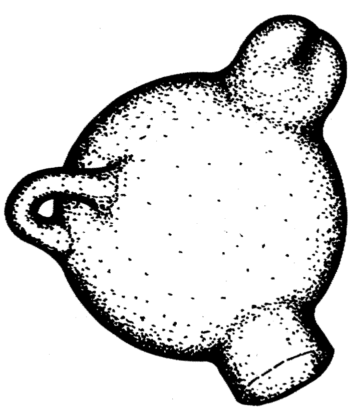


Fig. 1

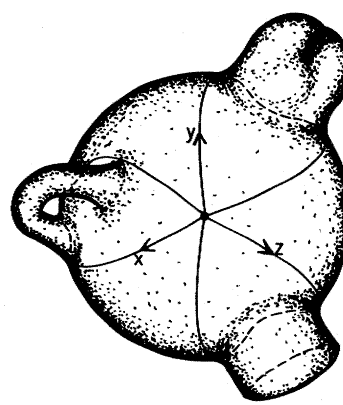


Fig. 2

*This paper is based on lectures given by Professor A. W. Tucker of Princeton University while a Philips visitor at Haverford College in the fall semester 1953-54. Microfilms of the complete notes of these lectures and some related material can be obtained from University Microfilms, 313 N. First St., Ann Arbor, Michigan (Cost: $2.70).

the surface consists of four pieces, three containing the appendages and the fourth the remaining portion of the sphere. These pieces will be studied individually.

The appendage cut off by the cut x is called a "handle". It can be continuously deformed into a torus with a hole, which is shown in Figure 3. After making the cuts a and b shown in Figure 4, the surface can be unfolded as shown in Figure 5. With suitable stretching and shrinking, it can then be spread out to form the plane region of Figure 6.

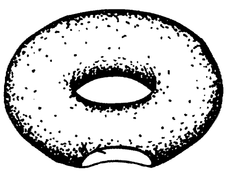


Fig. 3

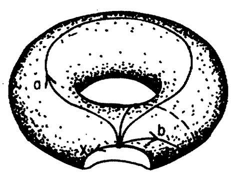


Fig. 4

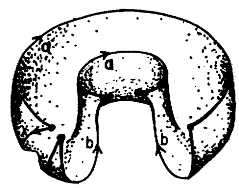


Fig. 5

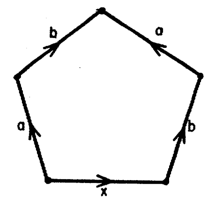


Fig. 6

The appendage cut off by the cut y is called a *crosscap*. It can be regarded as a hemisphere which has first had its "cap" cut off, as shown in Figure 7. The cap is then distorted by pulling the right side of the back through the right side of the front as the front is pushed back. This causes the cap to cross itself, as indicated in Figure 8. In order to be able to put the cap back on the base, the top of the base is pinched together to join the points B_1 and B_2 as the single point B. After the pieces are rejoined, the surface

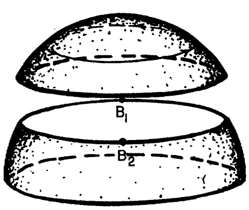


Fig. 7

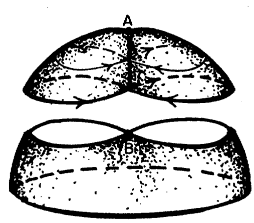


Fig. 8

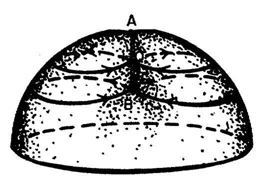


Fig. 9

is the crosscap of Figure 9. It crosses itself along AB. The points A and B are single points, but each point on the line between A and B is regarded as a double point - one on each of the portions of the surface crossing along AB. If one travels along the "figure eight" path through Q, he would go as indicated by the arrows in Figure 9. When moving from left to right through Q, he would not be aware of the other point at Q, which is on the section of this path which goes through Q from right to left. The crosscap can also be described as follows. Make a cut in a hemisphere and spread the cut apart as shown in Figure 10. Originally, the lines A_1B_1 and A_2B_1 were identified, and the lines A_1B_2 and A_2B_2 were identified (two lines being identified means that their corresponding points are regarded as being identical). Now identify the line A_1B_1 with the line A_2B_2 and the line A_2B_1 with the line A_1B_2. This forces A_1 and A_2 to be identified and B_1 and B_2 to be identified, again giving the crosscap of Figure 9. Now cut the crosscap along the path *c* and pull it through itself, as shown in Figure 11. The resulting surface

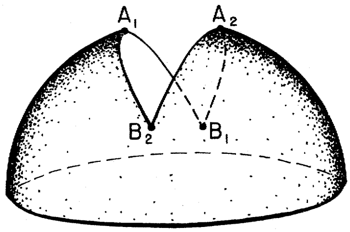


Fig. 10

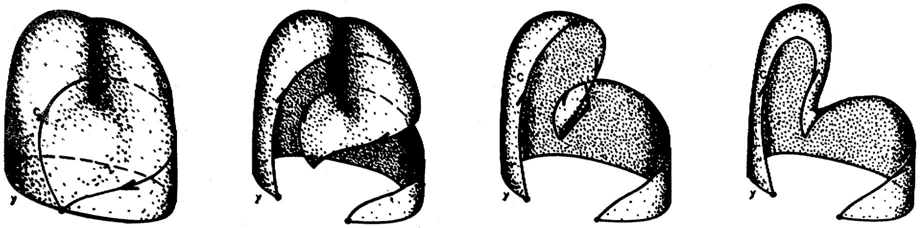


Fig. 11

can then be laid out as the triangle of Figure 12. A crosscap is topologically the same as a Moebius band in the sense that the points of the two surfaces can be put in a one-to-one correspondence that is continuous in both directions. In fact, if the Moebius band of Figure 13

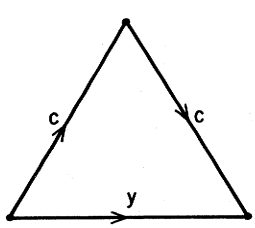


Fig. 12

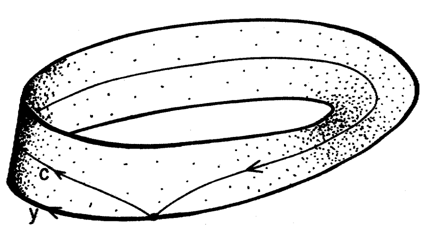


Fig. 13

is cut along the path *c* as indicated, it can be deformed into the triangle of Figure 12 and hence into a crosscap by reversing the process of Figures 11-12 [see Courant and Robbins, pages 260-262]. The surface shown in Figure 14 is also topologically the same as a Moebius band [see Tuckerman, *The American Mathematical Monthly*, vol. 55 (1948), pages 309–311]. It does not cross through itself and has a triangular boundary *ABC*. If this surface is cut along *CD*, *DG*, and *DEF*, it can be spread out as shown in Figure 15.

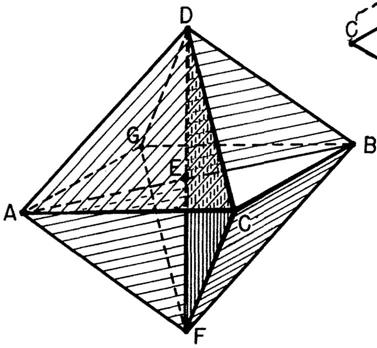


Fig. 14

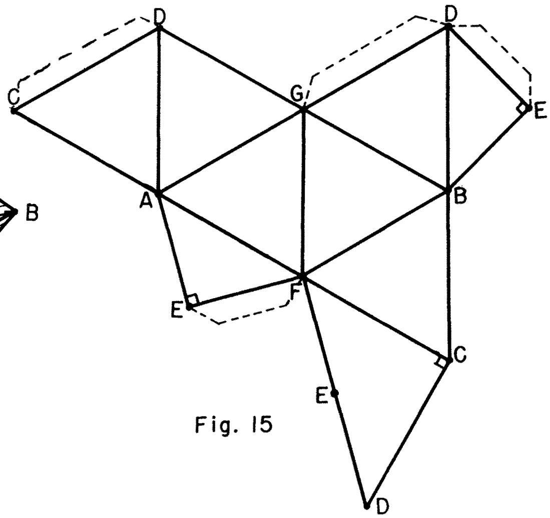


Fig. 15

This figure can be used as a pattern for constructing a model, with flaps as indicated to aid in rejoining corresponding edges. All folds are to be made upward, except for the flaps on *DE* and *EF*.

A *nonorientable surface* is one for which it is not possible to define orientation for circles in the surface in such a way that this orientation is preserved as a small circle is moved about in the surface. The Moebius band is a nonorientable surface. Figure 16 illustrates the fact that an oriented circle which is moved around a Moebius band returns to its original position with its orientation reversed. A nonorientable surface is also called a one-sided surface [Courant and Robbins, pages 259-264]. However, there is a technical objection to considering orientability and two-sidedness as synonomous (Orientability is an intrinsic property of the surface, but whether a surface is two-sided depends on the space in which it is embedded; e.g. a torus can be embedded in a certain 3-manifold so as to be one-sided).

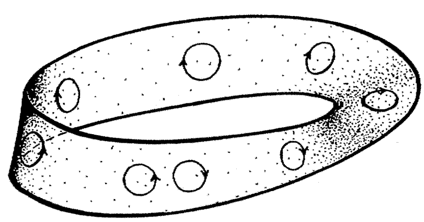


Fig. 16

The third appendage on the sphere of Figure 1 is the "cuff". If the free edge of the cuff is labeled as e and a cut d is made from the path z to the path e, the cuff of Figure 17 can be laid out as shown in Figure 18.

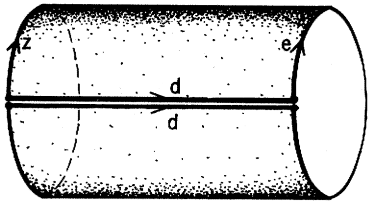


Fig. 17

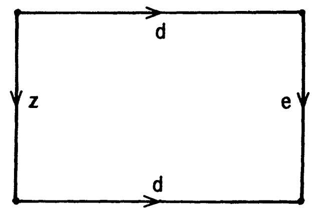


Fig. 18

The remainder of the sphere can now be stretched out as the triangle of Figure 19.

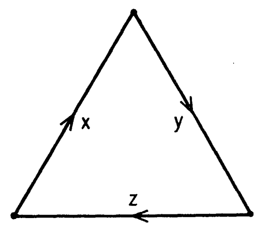


Fig. 19

The surface of Figure 1 has been cut into four pieces, each of which can be represented as a plane polygonal region. This piecewise description of the surface is conceptually very important because it is purely two-dimensional. The embedding of the surface in three-dimensional space has been eliminated. But the cutting into pieces is reversible. The pieces can be rejoined by pasting together edges labeled with the same symbol in such a way that directions correspond. Any two surfaces obtained in this way from the same pieces will be said to be combinatorially equivalent (see Definition 3.1). For example, the surfaces of Figures 9,13, and 14 are the results of three geometrically different ways of pasting together (identifying) the two edges labeled c in Figure 12, but these three surfaces are combinatorially equivalent.

Each of the pieces of the surface of Figure 1 (Figures 6, 12, 18, 19) can be represented symbolically by describing the order of the paths and the orientation of each path. Thus by traveling clockwise around each piece and using the $^{-1}$ sign to indicate that a path is being traveled in the opposite direction to the arrow on that particular path, one can obtain the following presentation of the four parts of the surface:

Handle: $aba^{-1}b^{-1}x^{-1} = 1,$

Crosscap: $ccy^{-1} = 1,$

Cuff: $ded^{-1}z^{-1} = 1,$

Remainder: $xyz = 1.$

These four relations are merely symbolic descriptions of the four pieces into which the surface of Figure 1 has been cut. It is clear that the order one traces around a piece and the point at which one starts are immaterial. Thus $zde^{-1}d^{-1} = 1$ or $ed^{-1}z^{-1}d = 1$ describe the same "cuff" as the relation $ded^{-1}z^{-1} = 1$ given above. The 1 in the right member of a relation is a symbolic formality that suggests the possibility of other forms of the same relation; for example, by multiplying on the right by z, the cuff relation becomes $ded^{-1} = z$. Also, the relations describing the four pieces of Figure 1 can be written as:

Handle:	$x = aba^{-1}b^{-1}$,
Crosscap:	$y = cc$,
Cuff:	$z = ded^{-1}$,
Remainder:	$xyz = 1$.

Each of these relations represents a plane polygonal region, it being understood that whenever one symbol occurs twice the corresponding edges are to be identified (or pasted together) with the directions on the edges matching. Thus the four pieces of Figures 6, 12, 18, and 19 can be pasted together to form Figure 20. After the x, y, z edges have been pasted together, they can be eliminated. This produces the polygonal

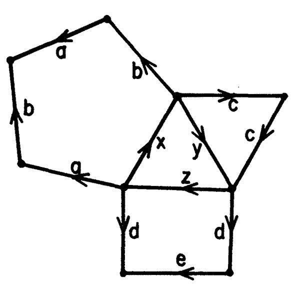


Fig. 20

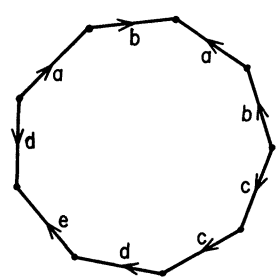


Fig. 21

region of Figure 21, which can be described by the single relation

$$aba^{-1}b^{-1}ccded^{-1} = 1.$$

This region represents the surface obtained by stretching and twisting the region, or even passing it through itself, in such a way that each pair of edges labeled with the same symbol (or a symbol and its inverse) are joined together with the directions on the edges matching. This surface will be said to be "equivalent in the sense of Combinatorial Topology", or *combinatorially equivalent*, to the surface of Figure 1. Since all cuts have been pasted together in the same way they were joined before cutting, and all other deformations are merely stretchings and shrinkings, it follows that the points of these two surfaces can be put in a one-to-one correspondence that is continuous in both directions [see Courant and Robbins, pages 241-243]. In such *point-set* terms, the two surfaces are called homeormorphic [see Newman, Chapter III].

Example 1.2. The above procedures will now be used to determine the topological nature of a particular interwoven covering surface of a sphere. This surface consists of two spheres (as indicated in Figure 22) which have been cut and rejoined so that the outer sheet of the Western Hemisphere joins the inner sheet of the Eastern Hemisphere along b_1 and d_1, and the inner sheet of the Western Hemisphere joins the outer sheet of the Eastern Hemisphere along b_2 and d_2. The resulting surface crosses itself along the curves A_1B_1 and A_2B_2. The points A_1, B_1, A_2, B_2 are single points, but the curve A_1B_1 represents the two distinct superimposed paths b_1 and b_2, and the curve A_2B_2 represents the two distinct superimposed paths d_1 and d_2.

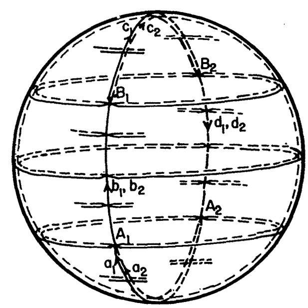


Fig. 22

Now cut the surface along the paths a_1, a_2; b_1, b_2; c_1, c_2; d_1, d_2. After the cutting, the surface will be in four pieces, which can be represented by the four polygonal regions of Figure 23. The first two of these were originally Western Hemispheres and the other two Eastern Hemispheres.

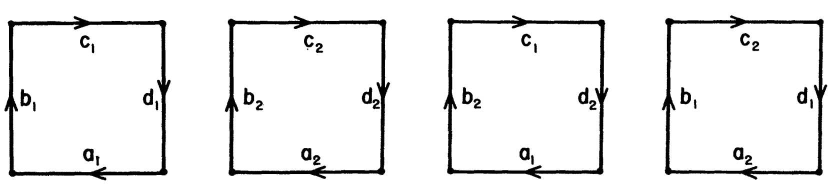


Fig. 23

Eliminate d_1 and d_2 by pasting together the edges labeled d_1 and pasting together the edges labeled d_2. This produces the two polygonal regions of Figure 24.

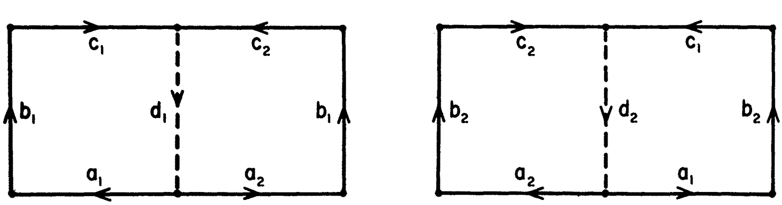


Fig. 24

Next paste these two regions together along $a_1^{-1}\,a_2$. After doing this, let $b_1^{-1}\,b_2 = \alpha$ and $c_2c_1^{-1} = \beta$. This produces the polygonal regions of Figures 25 and 26. Figure 27 shows that this polygonal region represents a torus. Thus the original interwoven covering surface of a sphere has been shown to be "equivalent" to a torus.

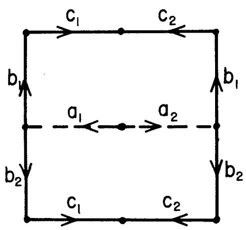


Fig. 25

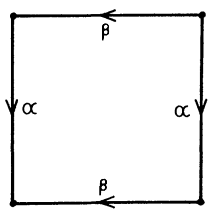


Fig. 26

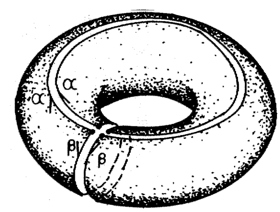


Fig. 27

The sequence of steps shown in Figure 28 illustrates how a torus and the surface of Fxample 1.2 can be continuously deformed into each other. The relation used to define the torus of Figure 28 is

$$(a_1a_2)(b_1b_2)(a_1a_2)^{-1}(b_1b_2)^{-1} = 1.$$

Care should be taken to imagine paths labeled with the same symbol as identified so that directions match and to note that the surface crosses itself along the curves A_1B_1 and A_2B_2.

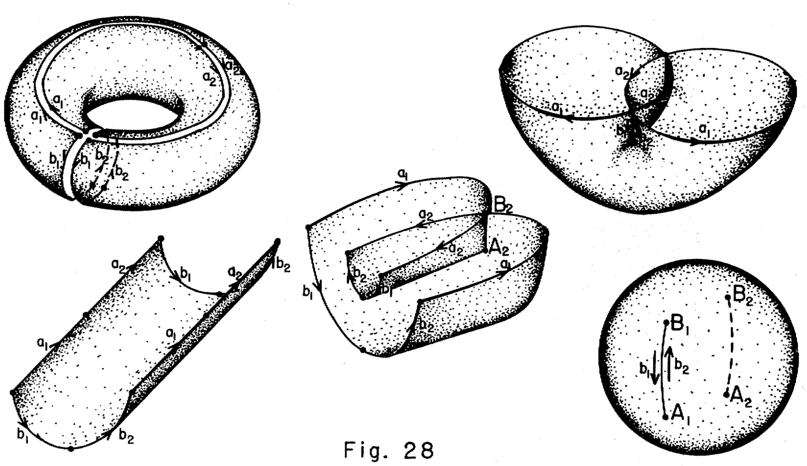


Fig. 28

The surface discussed in Example 1.2 is actually the Riemann surface which uniformizes the following function (also discussed by Curtiss in *Carus Mathematical Monographs No. 2*, page 167):

$$w^2 = (1 - z^2)(1 - k^2 z^2).$$

Each complex value of z except $z = \pm 1$ and $z = \pm 1/k$ determines two values of w. In order to get a continuous single-valued function of z, one can introduce a pair of z-planes (or sheets) and regard each value of z as determining exactly one value of w. These two sheets are connected, since for $z = \pm 1$ and $z = \pm 1/k$ there is only one value for w, namely $w = 0$. These four points are called branch points. The surface can be formed as indicated in Figure 29, as two sheets covering the z-plane and passing through each other along the dashed lines in the

Fig. 29

same way the crosscap of Figure 9 crosses itself (also see Example 5.2). The complex plane can be mapped onto a sphere by stereographic projection [see Hilbert and Cohn-Vossen, pages 248-259]. The Riemann surface then becomes the interwoven surface illustrated in Figure 22.

The surface of Figure 22 was cut into the four pieces represented by the polygonal regions of Figure 23. These regions can be described symbolically by the relations given in the following table.

	Western Hemisphere	*Eastern Hemisphere*
Outer Sheet	$a_1 b_1 c_1 d_1 = 1$	$a_1 b_2 c_1 d_2 = 1$
Inner Sheet	$a_2 b_2 c_2 d_2 = 1$	$a_2 b_1 c_2 d_1 = 1$

The pasting operations which were used to assemble these pieces to form a torus correspond to symbolic operations which will now be used to obtain the relation $\alpha\beta\alpha^{-1}\beta^{-1} = 1$ which describes a torus. First, write the relations for the two pieces covering the Eastern Hemisphere as

$$d_2 = c_1^{-1} b_2^{-1} a_1^{-1} \quad \text{and} \quad d_1 = c_2^{-1} b_1^{-1} a_2^{-1}.$$

Use these relations to replace d_1 and d_2 in the relations for the two pieces covering the Western Hemisphere, to get:

$$a_1 b_1 c_1 c_2^{-1} b_1^{-1} a_2^{-1} = 1, \quad a_2 b_2 c_2 c_1^{-1} b_2^{-1} a_1^{-1} = 1.$$

Now rewrite these two relations as follows:

$$b_1c_1c_2^{-1}\,b_1^{-1} = a_1^{-1}\,a_2, \qquad b_2c_2c_1^{-1}\,b_2^{-1}\,a_1^{-1}\,a_2^{-1} = 1.$$

Substitute the expression for $a_1^{-1}a_2$ into the second relation, giving:

$$b_2c_2c_1^{-1}\,b_2^{-1}\,b_1c_1c_2^{-1}\,b_1^{-1} = 1 \text{ or } b_1^{-1}\,b_2c_2c_1^{-1}(\,b_1^{-1}\,b_2)^{-1}(c_2c_1^{-1})^{-1} = 1.$$

With the substitutions $b_1^{-1}\,b_2 = \alpha$ and $c_2c_1^{-1} = \beta$, this relation becomes:

$$\alpha\beta\alpha^{-1}\beta^{-1} = 1,$$

which represents the torus of Figure 27.

2. *Surfaces defined by use of systems of relations.*

Our examples suggest that any surface might be represented symbolically by a finite system of relations in a finite number of symbols for which each symbol a has an associated symbol a^{-1}, called the inverse of a. These relations are statements equating two expressions, each of which is either an indicated product of symbols and inverses of symbols or simply 1 (1 is not one of the symbols). They are called relations to emphasize the difference between them and formal identities such as $(a^{-1})^{-1} = a$ or $a\cdot a^{-1} = 1$ which are used to manipulate relations (see Definitions 2.1 and 3.1). There are certain operations on a relation which do not in any way change the polygonal region which the relations can be interpreted as describing. These are given in the following definition.

Definition 2.1. Two systems of relations are *equivalent* if the relations of one system can be changed into the relations of the other by use of the following operations:

1) If x is a new symbol, then any symbol a can be replaced by x and a^{-1} by x^{-1}, provided this is done wherever a or a^{-1} appears.

2) Wherever it appears, any one of the expressions a, $1\cdot a$, $a\cdot 1$, or $(a^{-1})^{-1}$ can be replaced by any other one (these expressions are to be regarded as being merely different ways of writing the symbol a).

3) The last symbol on the right (left) of one member of a relation can be removed from this position if its inverse is put on the right (left) of the other member of the relation. This will be called *transposition* of the symbol.

For illustrations of the process of deriving equivalent relations from a given relation, consider the relation $ccy^{-1} = 1$ which describes the triangular region of Figure 12. This can be changed into $xxy^{-1} = 1$

by use of (1) of Definition 2.1. Or by use of (3), it can be changed into $cc = 1\cdot(y^{-1})^{-1}$ and then into $cc = y$ by use of (2). This last relation can be changed into $cc = y\cdot 1$ by use of (2) and then into $y^{-1}cc = 1$ by use of (3). By use of (3) and (2), one can change $cc = y$ into $c = yc^{-1}$, $1\cdot c = yc^{-1}$, and $1 = yc^{-1}c^{-1}$. Clearly these relations all describe the same triangular region, the differences being essentially differences in the starting point, the direction in which one moves around the boundary of the region, and the letters used to designate sides of the triangle.

A system of relations can be thought of as a *combinatorial presentation of a surface* and two equivalent systems of relations as different presentations of the same surface. The totality of systems equivalent to a given system of relations is an *equivalence class* which serves to define the surface in the sense of Combinatorial Topology. This is stated formally in the following definition. It should be noted that if (a) and (b) of this definition are satisfied by one system of relations, then they are satisfied by any equivalent system.

Definition 2.2. A *surface* is an object associated with an equivalence class of systems of relations for which each system consists of a finite number of relations in a finite number of symbols for which:

a) In the entire system of relations, no symbol occurs more than twice (either the symbol or its inverse is said to be an occurrence of the symbol).

b) If the system of relations is partitioned in any way into two systems (so that each relation belongs to exactly one of the two systems), then there is at least one symbol which occurs once in each of these two systems.

This definition can be given geometric meaning by interpreting each relation as describing a plane polygonal region and the set of relations as describing the surface obtained by identifying edges labeled with the same symbol so that directions correspond.

It should be noted that (a) and (b) of Definition 2.2 are satisfied by the systems of relations for the surfaces of Examples 1.1 and 1.2. The system of relations for Example 1.1 contains some symbols once and some twice, but no symbol occurs more than twice. Since the relation $xyz = 1$ has a symbol in common with each of the other relations, it follows that for any partition of the system of relations into two disjoint systems there is some symbol which occurs in each system. Any one of the systems of relations used to describe the surface of Example 1.2 contains each of its symbols exactly twice. The first system of four relations given for this surface were:

$$a_1b_1c_1d_1 = 1,\quad a_1b_2c_1d_2 = 1,\quad a_2b_2c_2d_2 = 1,\quad a_2b_1c_2d_1 = 1.$$

When written in this order, each adjacent pair has a symbol in common and the relations can be thought of as joined in a chain.

When interpreted geometrically, condition (b) of Definition 2.2 is the requirement that the surface consist of one piece. For suppose that the system of relations were partitioned into two non-empty systems. Then each system of relations would define a surface and these surfaces would be entirely disjoint if they have no symbols (edges) in common. It will be shown in Section 4 that any surface is combinatorially equivalent to a surface whose system of relations consists of a single relation. However, in addition to objects consisting of disjoint pieces, there are other objects which are not surfaces in the sense of Definition 2.2. For example, if a torus is squeezed to a point (P in Figure 30), the point is a conical point. Such an object can not be described by a set of relations which are to be interpreted geometrically as has been

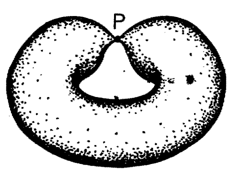


Fig. 30

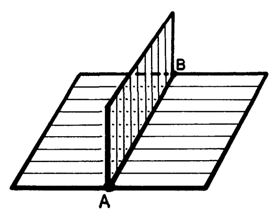


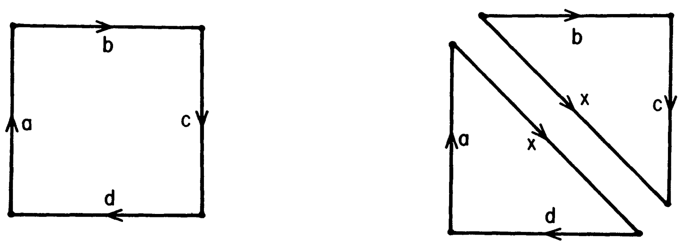
Fig. 31

done, since repeated symbols represent edges to be identified and there is no means for matching points alone. Also, the restriction (a) that no symbol occur more than twice prevents an edge from belonging to more than two faces (as for AB in Figure 31). The above ideas are consistent with the following definition of surface, as might be given by a point-set topologist: "A *closed surface* is a connected, compact metric space which is homogeneous in the sense that each point has a neighborhood which is a two-cell" [see Lefschetz, page 72]. A *surface with boundary curves* would then be defined by changing the neighborhood restriction so that each point on a boundary curve has a neighborhood which is half of a two-cell with the diameter included and lying along the boundary curve. As will be seen in Section 4, Definition 2.2 defines a surface which may have cuffs (and therefore boundary curves). It will be seen (Definition 4.3) that cuffs are present whenever some symbol occurs only once. If a system of relations has the property that each symbol occurs exactly twice, then any equivalent system also has this property (see Definition 2.1). Hence it is reasonable to define a closed surface as follows:

Definition 2.3. A *closed surface* is a surface which has a system of relations in which each symbol occurs exactly twice.

3. *Combinatorial equivalence of surfaces.*

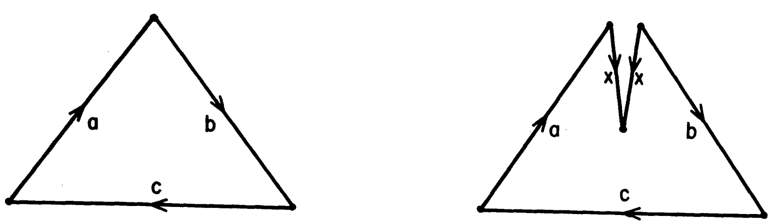
Let the system of relations for a surface be given geometric meaning by interpreting each relation as describing a plane polygonal region and the system of relations as describing the object obtained by identifying edges labeled with the same symbol so that directions correspond. This "geometric surface" is changed into a topologically equivalent "surface" by certain "cut-and-paste" operations which have corresponding symbolic operations. There are essentially three separate and distinct cutting operations, each of which has an inverse, a pasting operation. These six operations are enumerated below, with pictures illustrating the changes effected.



α: "Cut a face in two" (before the cut, or after the paste: $abcd=1$)

α': "Paste two faces together along a common edge" (after the cut, or before the paste: $axd=1$; $bc=x$)

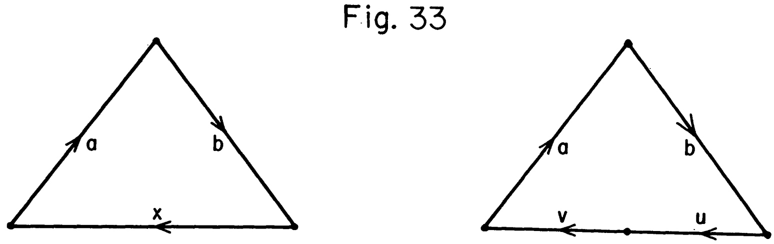
Fig. 32



β: "Cut part way into a face" (before the cut, or after the paste: $abc=1$)

β': "Paste two appropriate edges together" (after the cut, or before the paste: $axx^{-1}bc=1$)

Fig. 33



γ: "Break an edge in two (put in a new vertex)" (before the break, or after the weld: $abx=1$)

γ': "Weld two appropriate consecutive edges into a single one" (after the break, or before the weld: $abuv=1$)

Fig. 34

In order to classify the surfaces of Definition 2.2, it is useful to have some concept of equivalence of surfaces. This is given by the following definition. In this definition, capital letters represent blocks consisting of indicated products of symbols and inverses of symbols (or of 1 alone) and lower case letters are single symbols. If A is such a block, then A^{-1} represents the symbols of A in reverse order, with each symbol replaced by its inverse (e.g. if $A = ab^{-1}c$, then $A^{-1} = c^{-1}ba^{-1}$).

Definition 3.1. Two surfaces are *combinatorially equivalent* if a system of relations for one surface can be changed into a system of relations for the other surface by use of the following operations:

1) If x is a new symbol, then any one relation of form $ABC = 1$ can be replaced by the two relations $AxC = 1$, $B = x$.

1′) Any two relations of form $AxC = 1$, $B = x$ can be replaced by the single relation $ABC = 1$.

2) If x is a new symbol, then 1 can be replaced by xx^{-1}, or by $x^{-1}x$, in any one place.

2′) If x and x^{-1} appear side by side in any relation, their product can be replaced by 1.

3) If u and v are new symbols, then any symbol x can be replaced by uv, or x^{-1} by $v^{-1}u^{-1}$, provided this is done wherever x or x^{-1} appears.

3′) If x is a new symbol and the symbols u and v occur only in blocks uv or $v^{-1}u^{-1}$, then uv can be replaced by x and $v^{-1}u^{-1}$ by x^{-1}, provided this is done wherever uv or $v^{-1}u^{-1}$ appears.

The operations (1), (2), (3) of this definition correspond to the "cut" operations (α), (β), (γ). The operations (1′), (2′), (3′) correspond to the "paste" operations (α'), (β'), (γ'). Figures 32-34 illustrate special cases of these operations.

It should be noted that the indicated multiplication used in a relation is essentially non-commutative and associative. For the operations of Definition 3.1 do not permit interchange of ab and ba in general. But each member of a relation is regarded as a set of symbols written in a definite order with certain operations permitted with individual symbols or adjacent symbols. Thus the order of the symbols is important, but the particular method of grouping into blocks which may be used is merely a notational convenience. It should also be noted that each operation of Definition 3.1 has an inverse operation. In fact, the relation of combinatorial equivalence is an equivalence relation.

4. *Canonical Forms and Classification of Surfaces.*

Before showing how to reduce a system of relations to canonical form, two transformation rules will be developed. First consider a relation in which a block Q of symbols is adjacent to a symbol x which occurs twice, both times as x rather than x^{-1}. Such a relation can be written as

$$AxQBxC = 1, \quad \text{or} \quad AQxBxC = 1,$$

where capital letters represent either 1 or blocks consisting of products of symbols and inverses of symbols. Use (1) of Definition 3.1 and a new symbol y to replace these relations by the two pairs of relations:

$$\begin{cases} AyBxC = 1, \\ xQ = y; \end{cases} \qquad \begin{cases} AyBxC = 1, \\ Qx = y. \end{cases}$$

By repeated transpositions of symbols (Definition 2.1), $xQ = y$ and $Qx = y$ can be changed into $x = yQ^{-1}$ and $x = Q^{-1}y$, respectively, where Q^{-1} is the block consisting of the symbols of Q multiplied in reverse order with each symbol replaced by its inverse. Now use (1′) of Definition 3.1 to replace the above pairs of relations by the single relations

$$AyByQ^{-1}C = 1 \quad \text{and} \quad AyBQ^{-1}yC = 1.$$

Since x did not occur more than twice, it has now been completely eliminated and we can use (1) of Definition 2.1 to replace y by x. This gives:

$$AxBxQ^{-1}C = 1 \quad \text{and} \quad AxBQ^{-1}xC = 1.$$

RULE I: *If a symbol x occurs twice in the same relation, both times as x, then a block Q which is on one side of x in one position can be removed, provided Q^{-1} is put on the same side of x in the other position.*

The geometrical interpretation of this rule is shown in Figure 35. A cut has been made along y. Then the shaded piece has been moved and pasted along x, eliminating x. Then y is replaced by x.

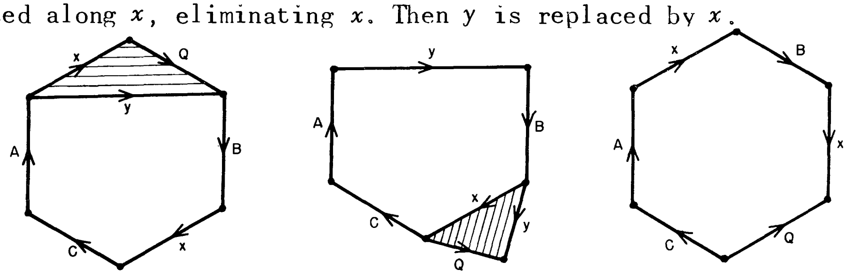


Fig. 35

The second rule is similar to the first. Consider a relation in which a block Q of symbols is adjacent to a symbol x which occurs once as x and once as x^{-1}. Such a relation can be written as

$$A\,x\,Q\,B\,x^{-1}C = 1 \quad \text{or} \quad A\,Q\,x\,B\,x^{-1}C = 1.$$

Again use is made of (1) of Definition 3.1 and a new symbol y to replace these relations by the two pairs of relations:

$$\begin{cases} A\,y\,B\,x^{-1}C = 1, \\ x\,Q = y; \end{cases} \quad \text{and} \quad \begin{cases} A\,y\,B\,x^{-1}C = 1, \\ Q\,x = y \end{cases}$$

Transpositions of the symbols x and y can be used to change $xQ = y$ and $Qx = y$ into $Qy^{-1} = x^{-1}$ and $y^{-1}Q = x^{-1}$, respectively. Use can now be made of (1′) of Definition 3.1 to replace the pairs of relations by the single relations:

$$A\,y\,B\,Q\,y^{-1}C = 1 \quad \text{and} \quad A\,y\,B\,y^{-1}\,Q\,C = 1.$$

Replacing y by x gives:

$$A\,x\,B\,Q\,x^{-1}C = 1 \quad \text{and} \quad A\,x\,B\,x^{-1}\,Q\,C = 1.$$

RULE II: *If a symbol x occurs twice in the same relation, once as as x and once as x^{-1}, then a block Q which is on one side of x can be moved (without inversion) to the other side of x^{-1}*

The geometrical interpretation of this rule is quite similar to that of Rule I, shown in Figure 35.

It should be carefully noted that the symbols of a relation are essentially dummy variables, since (1) of Definition 2.1 enables one to replace a symbol and its inverse by a new symbol and its inverse whenever desired. Rules I and II do not merely move symbols around, but change the meaning of some of them. However, the new system of relations defines a surface which is combinatorially equivalent to the original surface.

These two rules are the principal tools used in the following proof that a system of relations for any surface (in the sense of Definition 2.2) can be reduced to one of four types of canonical forms by use of the transformations permitted by Definition 3.1. The process of reduction to canonical form is outlined in the following series of five steps.

STEP 1. Reduce the system of relations to a single relation. Choose any one of the relations of a system of relations which consists of two or more relations. Then it follows from (b) of Definition 2.2 that there is another relation and a symbol x such that x occurs in each of these relations. By repeated transpositions, these two relations can be written in the forms: $A\,x = 1$, $B = x$. They can then be replaced by

$AB = 1$, using (1′) of Definition 3.1. Any symbol which occurs in $AB = 1$ also occurs in either $Ax = 1$ or $B = x$. Also, any symbol which occurs in one of these two relations and in some third relation also occurs in $AB = 1$. Thus the new set of relations also satisfies (b) of Definition 2.2. Since the system contains a finite number of relations, this process will eventually reduce it to a single relation.

STEP 2. Assemble the crosscaps. Suppose c is any symbol which occurs twice, both times as c rather than c^{-1}. By transposing symbols (if necessary), the relation can be written in the form

$$ABcCcD = 1,$$

where (as before) the capital letters represent blocks of symbols (or simply 1). Geometrically, this relation can be interpreted as representing the polygonal region of Figure 36.

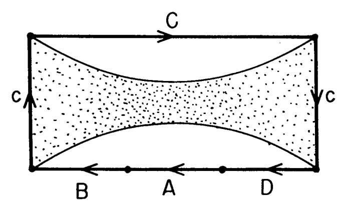


Fig. 36

If this figure is cut along the dashed lines and the edges labeled c pasted together with the directions matching, the shaded region becomes a Moebius band (Figure 13). Now use Rule I to change this relation into $AcCB^{-1}cD = 1$ and then into $AccBC^{-1}D = 1$. If c_1 is a symbol which occurs twice as c_1, then this process with A taken as 1 produces a relation of form $c_1c_1P = 1$. If some other symbol c_2 now occurs twice as c_2, the same process can be used with A representing c_1c_1. This changes the relation into the form $c_1c_1c_2c_2P' = 1$. This can be continued until the relation is of the form $c_1c_1 \ldots c_qc_qQ = 1$, where any symbol x which occurs twice in Q occurs once as x and once as x^{-1}. Now use (1) of Definition 3.1 to replace each c_ic_i by a new symbol y_i and introduce the new relations $y_i = c_ic_i$. We now have a system of relations of the following form:

$$\begin{cases} y_i = c_ic_i, \quad i = 1, 2, \ldots, q; \\ y_1 \ldots y_qQ = 1. \end{cases}$$

STEP 3. Assemble the handles. Let a and b be any two symbols which occur twice in such a way that the relation $y_1 \ldots y_qQ = 1$ is of the form:

$$ABaCbDa^{-1}Eb^{-1}F = 1.$$

By repeated use of Rule II, this relation can be successively changed into the following relations:

$$AaCbDa^{-1}(BE)b^{-1}F = 1,$$

$$AaCb(BED)a^{-1}b^{-1}F = 1,$$

$$Aa(BEDC)ba^{-1}b^{-1}F = 1,$$

$$Aaba^{-1}b^{-1}(BEDCF) = 1.$$

If a_1 and b_1 are two symbols which occur in the same way as a and b above and A represents 1, this process changes $y_1 \cdots y_q Q = 1$ into the form:

$$a_1 b_1 a_1^{-1} b_1^{-1} y_1 \cdots y_q Q' = 1.$$

If two other symbols a_2 and b_2 occur in the same way as a and b and A represents $a_1 b_1 a_1^{-1} b_1^{-1}$, the result is of the form:

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} y_1 \cdots y_q Q'' = 1.$$

This can be continued until the relation is of the form:

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_p b_p a_p^{-1} b_p^{-1} y_1 \cdots y_q R = 1,$$

where no symbol which occurs twice in R can occur exactly once between two occurrences of another symbol. Now use (1) of Definition 3.1 to replace each $a_i b_i a_i^{-1} b_i^{-1}$ by a new symbol x_i and introduce the new relation $x_i = a_i b_i a_i^{-1} b_i^{-1}$. We now have a system of relations of the following form:

$$\begin{cases} x_i = a_i b_i a_i^{-1} b_i^{-1}, & i = 1, \ldots, p; \\ y_i = c_i c_i, & i = 1, \ldots, q; \\ x_1 \cdots x_p y_1 \cdots y_q R = 1. \end{cases}$$

STEP 4. Assemble the cuffs. If there is a symbol which occurs twice in the block R of the last relation above, choose a symbol d for which there are as few symbols between d and d^{-1} as for any other such paired symbols. If there are no symbols between d and d^{-1}, then $d\,d^{-1}$ can be removed by operation (2′) of Definition 3.1. Otherwise, the only symbols which can be between d and d^{-1} are symbols which occur only once (for if y and y^{-1} were both between d and d^{-1}, there would be fewer symbols between y and y^{-1} than between d and d^{-1}; while if y^{-1} were not between d and d^{-1}, another handle would have been assembled). Since the symbols between d and d^{-1} occur only once, several applications of (3′) of Definition 3.1 can be used to replace this block of symbols by a single new symbol e. The block R is now of the form $Aded^{-1}B$. By use of Rule II, this can be changed into $ded^{-1}AB$. By continuing this process, R can finally be written as:

$$d_1e_1d_1^{-1}\ d_2e_2d_2^{-1}\ \ldots d_re_rd_r^{-1}S,$$

where S is either 1 or consists entirely of symbols which occur only once. If S is not 1, replace S by a new symbol e. Also use (1) of Definition 3.1 to replace $d_ie_id_i^{-1}$ by z_i for each i and introduce the new relations $z_i = d_ie_id_i^{-1}$. The system of relations is now:

$$\begin{cases} x_i = a_ib_ia_i^{-1}b_i^{-1}, & i = 1, \cdots, p; \\ y_i = c_ic_i, & i = 1, \cdots, q; \\ z_i = d_ie_id_i^{-1}, & i = 1, \cdots, r; \\ x_1 \cdots x_py_1 \cdots y_qz_1 \cdots z_re = 1. \end{cases}$$

If one thinks of the geometric interpretation of these relations, it can be seen that each x_i, y_i, and z_i represents a path whose initial and terminal points are identified. Let W be the path $x_1 \cdots x_py_1 \cdots y_qz_1 \cdots z_r$. Then W also has its initial and terminal points identified and We would seem (intuitively) to be equivalent to $Wded^{-1}$, as shown in Figure 37. This change can be interpreted as representing a shift in

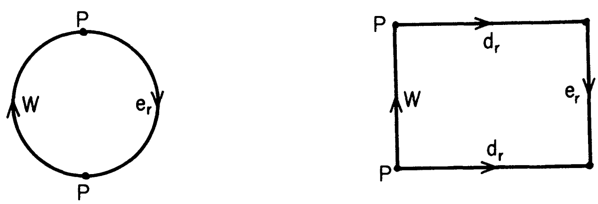


Fig. 37

the point P in a dissection of a surface for which P originally was on the free edge e of a cuff. The introduction of d can be formally justified as follows. Replace the relation $We = 1$ by $Wd^{-1}de = 1$, using (2) of Definition 2.1 and (2) of Definition 3.1. Then d^{-1} can be moved to the left past each z_i by use of Rule II (replacing z_i by $d_ie_id_i^{-1}$ before the use of the rule and putting z_i back after using the rule). Simmilarly, d^{-1} can be moved past each y_i by two applications of Rule I. It can then be moved past each x_i by four applications of Rule II. The relation now is $d^{-1}Wde = 1$, which can be changed into $Wded^{-1} = 1$ by two transpositions. Let r be increased by 1 and z_r be a new symbol. Replace the relation $Wded^{-1} = 1$ by $Wz_r = 1$ and $z_r = d_re_rd_r^{-1}$. We now have a system of relations of the form:

$$\begin{cases} x_i = a_i b_i a_i^{-1} b_i^{-1}, & i = 1, \ldots, p; \\ y_i = c_i c_i, & i = 1, \ldots, q; \\ z_i = d_i e_i d_i^{-1}, & i = 1, \ldots, r; \\ x_1 \ldots x_p y_1 \ldots y_q z_1 \ldots z_r = 1. \end{cases}$$

STEP 5. Turn the handles into crosscaps (if there is at least one crosscap). It will be seen that handles can not be turned into crosscaps unless there is at least one crosscap to work with. Thus the original crosscap is a sort of "catalyzer" in the process. Consider a block $aba^{-1}b^{-1}cc$ Apply Rule I as follows: first to the symbol c and the block $a^{-1}b^{-1}$, giving $abcbac$; then to the symbol a and the block bcb, giving $aab^{-1}c^{-1}b^{-1}c$; and finally to the symbol b^{-1} and the block c^{-1}, giving $aab^{-1}b^{-1}cc$. Thus the handle $x_p = a_p b_p a_p^{-1} b_p^{-1}$ and the crosscap $y_1 = c_1 c_1$ can be substituted into the relation $x_1 \ldots x_p y_1 \ldots y_q z_1 \ldots z_r = 1$, made into three crosscaps, and the relations put back in the above form with q larger by 2 and p smaller by 1. This can be continued until $p = 0$.

By use of the above five steps, the system of relations for a surface can be reduced to a canonical form which is of one of the following four general types:

CANONICAL FORMS

1. *Closed orientable surface.* The surface has no crosscaps and no cuffs ($q = r = 0$) and is said to be of genus p ($p \geq 0$).

$$\begin{cases} x_i = a_i b_i a_i^{-1} b_i^{-1}, & i = 1, \ldots, p; \\ x_1 \ldots x_p = 1. \end{cases}$$

2. *Orientable surface with cuffs.* The surface has no crosscaps ($q = 0$), but at least one cuff ($r > 0$).

$$\begin{cases} x_i = a_i b_i a_i^{-1} b_i^{-1}, & i = 1, \ldots, p; \\ z_i = d_i e_i d_i^{-1}, & i = 1, \ldots, r; \\ x_1 \ldots x_p z_1 \ldots z_r = 1. \end{cases}$$

3. *Closed nonorientable surface.* The surface has no handles and no boundaries ($p = r = 0$), but at least one crosscap ($q > 0$).

$$\begin{cases} y_i = c_i c_i, & i = 1, \ldots, q; \\ y_1 \ldots y_q = 1. \end{cases}$$

4. *Nonorientable surface with cuffs*. The surface has no handles ($p = 0$), but at least one crosscap ($q > 0$) and at least one cuff ($r > 0$).

$$\begin{cases} y_i = c_i c_i, & i = 1, \ldots, q; \\ z_i = d_i e_i d_i^{-1}, & i = 1, \ldots, r; \\ y_1 \ldots y_q z_1 \ldots z_r = 1. \end{cases}$$

It should be noted that a surface with $p = q = r = 0$ is a closed orientable surface whose canonical form consists of the single relation $1 = 1$. Such a surface is combinatorially equivalent to a sphere, since a sphere is the geometrical interpretation of the relation $xx^{-1} = 1$.

It will be shown that the numbers p, q, and r are the same for two combinatorially equivalent surfaces whose systems of relations are in canonical form. The method for doing this will depend on procedures for determining: A) whether a surface is orientable; B) the number of cuffs; C) a certain number χ. These procedures will also be used to develop a method for identifying the canonical form for a surface without first reducing the system of relations to canonical form. This complete process will be illustrated by applying it to the example of Figure 38. This surface will be shown to be a nonorientable surface with two crosscaps and two cuffs. That is, a Klein bottle with two cuffs (as shown in Figure 39). The relations for this surface are:

$$abdfgm = 1, \quad ac^{-1}efhj = 1, \quad bcm^{-1}k^{-1}j = 1.$$

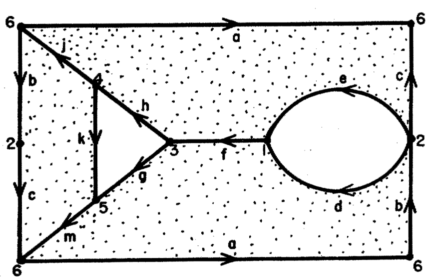


Fig. 38

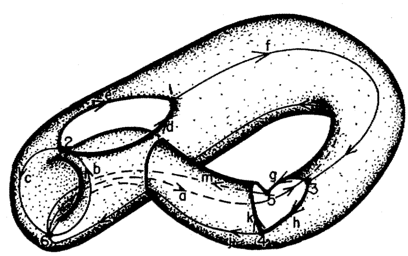


Fig. 39

(A) In the above testing of canonical forms, a surface whose system of relations is in canonical form was called orientable if $q = 0$; that is, if no "piece" of the surface is a Moebius band. This concept of orientability is intuitively equivalent to that discussed in connection with Figure 16. The following definition is an extension of the $q = 0$ test and enables one to determine the orientability of a surface from its relations even if they are not in canonical form.

Definition 4.1. A surface is *orientable* if it has a system of relations for which each relation is of the form $A = 1$ and any symbol x which occurs twice occurs once as x and once as x^{-1}.

The surface of Figure 38 is nonorientable. Because of the way c and j occur, there is no way of using the operations of Definition 2.1 to obtain relations in the form $A = 1$ for which no symbol occurs twice in the same form.

Showing that two combinatorially equivalent surfaces are either both orientable or both nonorientable is equivalent to showing that the property of orientability is preserved by the operations of Definition 3.1; that is, that a system of relations which is equivalent to a system of the type described in Definition 4.1 still has this property after any manipulation by the operations of Definition 3.1. Suppose $ABC=1$ is one of the relations in a system of relations for an orientable surface. Then this system of relations has the property of being equivalent to a system of relations for which $ABC=1$ is one of the relations and each symbol x which occurs twice occurs once as x and once as x^{-1} (see Definition 2.1). The system still has this property if operation (1) of Definition 3.1 is used to replace $ABC=1$ by the two relations $AxC = 1$, $B = x$, and then $B = x$ is changed into $Bx^{-1} = 1$. The inverse operation (1′) can be treated simmilarly. It is easy to see that the other operations of Definition 3.1 do not affect the orientability of the surface. Thus orientability is combinatorially invariant and a surface is orientable if and only if $q = 0$ for a canonical form of its system of relations, or if and only if no part of the surface is a Moebius band (see Figure 36).

Before discussing the *number of cuffs* for a surface whose system of relations is not in canonical form, it is convenient to introduce the concept of vertices.

Definition 4.2. Each symbol (in a system of relations for a surface) has two vertices. These are not necessarily distinct, but are called the *initial vertex* and the *terminal vertex*. The initial (terminal) vertex of a symbol x is the same as the terminal (initial) vertex of x^{-1}; if two symbols x and y can be made adjacent (in the order xy) by transposing symbols, then the terminal vertex of x is the same as the initial vertex of y; if x occurs in the relation $x = 1$, then the initial vertex of x is also the terminal vertex of x.

To determine the vertices for the surface of Figure 38, let P_1 be the terminal vertex of d. It is then the initial vertex of f and, finally, the terminal vertex of e. This is indicated in the relations below by putting 1 to the right of d; since 1 is to the left of f, it is put to the left of the other occurrence of f; it is then at the right of e; since e is unmatched, the chain is ended. This process can be continued to determine the other vertices. The surface has six vertices, as labeled in Figure 38.

$a\underline{6}b\underline{2}d\underline{1}f\underline{3}g\underline{5}m\underline{6} = 1,\quad a\underline{6}c^{-1}\underline{2}e\underline{1}f\underline{3}h\underline{4}j\underline{6} = 1,\quad b\underline{2}c\underline{6}m^{-1}\underline{5}k^{-1}\underline{4}j\underline{6} = 1.$

Notice that each of the vertices 1,2,3,4,5 of Figure 38 is on boundary edges and that the boundary edges and these boundary vertices fall naturally into the two cyclic sets $\underline{1}d^{-1}\underline{2}e$ and $\underline{3}h\underline{4}k\underline{5}g^{-1}$, which can be obtained by tracing around each closed curve in the boundary. This suggests the following definition and lemma.

Definition 4.3. A vertex is said to be a *boundary vertex* if it is a vertex of a symbol which occurs only once. All other vertices are said to be *interior vertices*. A symbol is said to be a *boundary edge* if it occurs only once and an *interior edge* if it occurs twice.

Lemma. each boundary vertex either is a single vertex of exactly two boundary edges or is both the initial and terminal vertex of one boundary edge and is not a vertex of any other boundary edge. The boundary edges can be arranged in cyclic sets so that any two adjacent boundary edges have a common vertex.

This lemma can be verified as follows. Note that if x is any symbol, then either x occurs only in the relation $x = 1$, or there is a unique symbol which can (by transpositions of symbols) be made adjacent to x on the right and there is a unique symbol which can be made adjacent to x on the left. Hence if P is the terminal (initial) vertex of an unmatched symbol (boundary edge) x, then P is also an initial (terminal) vertex of some symbol y_1(possibly y_1 is x). If y_1 occurs twice, then P is a vertex of a symbol y_2 adjacent to y_1 at the other occurrence of y_1. Continuing this process, it can be seen that it terminates as soon as one of the y_i is an unmatched symbol (possibly x).

(B). If the system of relations for a surface is in canonical form, then the number of cuffs is the number r. To show that the number of cuffs is uniquely defined for any given surface, it will be shown that the number of cuffs (as defined below) is the same for any two combinatorially equivalent surfaces and is equal to r if the set of relations is in canonical form.

Definition 4.4. Given a system of relations for a surface, the *number of cuffs* on the surface is the largest number of sets into which the unmatched symbols (boundary edges) can be divided with the restriction that two symbols must belong to the same set if they have a vertex in common.

It follows from the above lemma that the sets of unmatched symbols of Definition 4.4 must be cyclic sets for which any two adjacent symbols have a common vertex. For the surface of Figure 38, these sets are (d, e), with vertices (1,2), and (g, h, k), with vertices (3,4,5). Thus the surface has two cuffs and it is expected that, if the relations were put in canonical form, r would equal 2.

It will now be shown that the number of cuffs is invariant under the operations of Definition 3.1. For operations (1) or (1′) and the relations $ABC = 1$ and $B = x$, the relation $B = x$ can be written as $x^{-1}B = 1$, or as $Bx^{-1} = 1$. Therefore the terminal vertex of A is the initial vertex of B and the initial vertex of x. The initial vertex of C is the terminal vertex of B and the terminal vertex of x. Thus each vertex of x is a vertex for some other symbol (see Figure 32). Since x occurs twice, there is no change in the set of unmatched symbols and no change in their vertices. This is also true of operations (2) and (2'), although these operations create (or destroy) one vertex, which is either the terminal vertex of x and the initial vertex of x^{-1} (when xx^{-1} occurs) or the initial vertex of x and the terminal vertex of x^{-1} (when $x^{-1}x$ occurs), but is not a vertex for any other symbol (see Figure 33). Operations (3) and (3') also create (or destroy) a vertex, which is the terminal vertex of u and the initial vertex of v, but is not a vertex of any other symbol. If u and v are unmatched symbols, then the sets of unmatched symbols are changed only by replacing x by both u and v (or u and v by x), since the initial vertex of x is the initial vertex of u and the terminal vertex of x is the terminal vertex of v.

For a set of relations in canonical form, the unmatched symbols are $e_1 \dots e_r$. For each e_i, the initial vertex is also the terminal vertex. It is also the terminal vertex of d_i and the initial vertex of d_i^{-1}, but is not a vertex of any other symbol. Thus two different symbols e_i and e_j do not have a common vertex, and the number of cuffs is equal to r.

(C). For a surface with a system of relations in canonical form, the number $\chi = 2-2p-q-r$ is called the *Euler characteristic* of the surface [the meaning of χ for closed orientable surfaces is discussed by Courant and Robbins, pages 236-240, 258-259]. The following definition of χ has meaning for any surface in the sense of Definition 2.2. It will be shown that two combinatorially equivalent surfaces have the same value for x and that $\chi = 2-2p-q-r$ for a surface with a system of relations in canonical form.

Definition 4.5. Given a system of relations for a surface, the *Euler characteristic* χ of the surface is the number $\chi = V-E+F$, where V is the number of vertices, E the number of symbols, and F the number of relations.

If the surface is given the geometrical interpretation used previously, then V is the number of vertices, E the number of edges or segments, and F the number of faces or regions. For the surface of Figure 38, $\chi = 6-11+3 = -2$.

As noted before, operation (1) of Definition 3.1 does not change the number of vertices; however, it increases the number of relations by 1 and increases the number of symbols by 1. It therefore does not change χ. Operation (2) does not change χ, since deleting xx^{-1} (or $x^{-1}x$) deletes the symbol x and the terminal (or initial) vertex of x. Operation (3) increases the number of symbols by 1, but also increases the number of vertices by 1. Operations (1′), (2′), and (3′) are the inverses of (1), (2), and (3) and therefore also do not change χ.

For a system of relations in canonical form, there are $(r+1)$ vertices: one vertex is both the initial and terminal vertex for each x_i, a_i, b_i, y_i, c_i, z_i, and also is the initial vertex for each d_i; the terminal vertex for d_i is the same as the initial and terminal vertices for e_i $(i = 1, \ldots, r)$. There are $3p + 2q + 3r$ symbols and $p + q + r + 1$ relations. Hence

$$\chi = r + 1 - (3p + 2q + 3r) + (p + q + r + 1) = 2 - 2p - q - r.$$

The property of orientability, the number of cuffs (r), and the number χ have now been shown to be the same for any two combinatorially equivalent surfaces. Knowing whether the surface is orientable or not (whether $q = 0$ or $q > 0$) and whether $r = 0$ is enough information to uniquely determine one of the four types of canonical forms. Then the values of χ and r are enough additional information to determine the values of p and q, since $\chi = 2 - 2p$, $\chi = 2 - 2p - r$, $\chi = 2 - q$, and $\chi = 2 - q - r$, respectively, in each of the four classifications of canonical forms.

For example, the surface of Figure 38 is now known to be nonorientable, to have $r = 2$ and $\chi = -2$. Therefore it is a nonorientable surface with cuffs. Since $\chi = 2 - q - r$, it follows that $q = 2$ and the canonical form is:

$$\begin{cases} y_i = c_i c_i, & i = 1, 2; \\ z_i = d_i e_i d_i^{-1}, & i = 1, 2; \\ y_1 y_2 z_1 z_2 = 1. \end{cases}$$

A surface has $\chi = 2$ if and only if $p = q = r = 0$. For such a surface, the system of relations in canonical form is $1 = 1$ and the surface is combinatorially equivalent to a sphere. However, any simple polyhedron is combinatorially equivalent to a sphere. Therefore $V - E + F = 2$ for any simple polyhedron. This is called *Euler's Formula* [see Courant and Robbins, page 236]. The number χ is called *Euler's characteristic,* although Euler only dealt with the case $\chi = 2$ of simple polyhedra (which can be continuously deformed into a sphere).

There are only 4 surfaces for which $\chi = 0$, for the only non-negative integral solutions of $0 = 2 - 2p - q - r$ are: (1,0,0), (0,0,2), (0,1,1), (0,2,0). These are, respectively, the torus (sphere with one handle), cylinder (sphere with two cuffs), Moebius band, and Klein bottle. These surfaces are shown in Figure 40. They are the only four surfaces which admit a comtinuous tangent vector field which is nowhere zero. Poincaré [Journal de Mathematique (4), v. 1 (1885), p. 203] showed that, for a

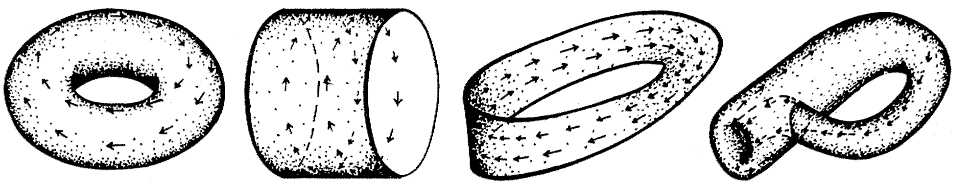


Fig. 40

continuous tangent vector field having at most a finite number of simple singularities (zeros), the Euler characteristic is equal to the difference in number between those singularities of positive "index" (nodes, foci) and those of negative "index" (saddle points) [see Lefschetz, Introduction to Topology, pages 17-19].

The Betti number and the connectivity number of a surface are combinatorially invariant numbers which are closely related to χ [see Tucker and Bailey]. The *Betti number B* is equal to $2-\chi$ for a closed surface and to $1-\chi$ for a surface with cuffs*. It is also the maximum number of cuts that can be made on the surface without dividing it into more than one piece, where: (1) the cuts are closed paths (or paths joining two points on previous cuts) if the surface is closed [see Hilbert and Cohn-Vossen, page 294]; (2) the cuts are along paths joining two points on free edges (boundaries) of cuffs if the surface is not closed. If a surface has *r* cuffs, then *B* is the maximum number of cuts along closed paths which can be made without getting more than *r* pieces. The idea of connectivity was introduced by Riemann in 1857 [see Smith, pages 404-410]. A surface is said to be *simply connected* if any closed curve in the surface can be continuously deformed to a point without leaving the surface (e.g. the disk and sphere are simply connected); otherwise, the surface is said to be multiply connected [see Courant and Robbins, pages 243-244]. The *connectivity number h* is equal to $1+B$, and is 1 for a simply connected surface.

	h	B	χ	p	q	r
disk	1	0	1	0	0	1
plane annulus	2	1	0	0	0	2
Moebius band	2	1	0	0	1	1
projective plane	2	1	1	0	1	0
torus	3	2	0	1	0	0
Klein bottle	3	2	0	0	2	0

* Strictly, this is the one-dimensional Betti number mod 2 [see Lefschetz, Introduction to Topology, pages 68-72, 99-103].

5. *Coverings of the Sphere*

Let R be a system of relations for a surface S, the symbols in R being denoted by letters $a, b, c, \ldots$. Replace these letters by a_i, b_i, c_i, ... ($i = 1, \ldots, n$). This makes n systems of relations $R_1, \ldots, R_n$ which define n copies $S_1, \ldots, S_n$ of the original surface S. Suppose a symbol x occurs twice in R, and that the second occurrence of x_1 in R_1 is replaced by x_2 and the second occurrence of x_2 in R_2 is replaced by x_1. These new systems of relations R_1' and R_2' together define the surface which can be thought of as being formed by cutting S_1 and S_2 along x_1 and x_2 and pasting the surfaces together by crossing them so that each side of x_1 is joined to the other side of x_2 (this is the process that was used in Example 1.2). More generally, for each x which occurs twice in R, one can permute the second occurrences (or the first occurrences) of $x_1, \ldots, x_n$ in $R_1, \ldots, R_n$. If this is done so that the new relations $R_1', \ldots, R_n'$ satisfy condition (b) of Definition 2.2 (i.e. so that all n sheets are connected), these relations together define a surface which is said to be an n-sheeted covering surface of S. For example, this can be done by a cyclic permutation of the second occurrences of one letter (the other letters being permuted in any way).

Note that a covering surface of a closed surface is closed, since the above process can not create unmatched symbols (see Definition 4.4). Since permutations are made among symbols $x_1, \ldots, x_n$ which occur in the same form (either as x_i or as x_i^{-1}), it follows that if the relations for S are written so that no symbol occurs twice in the same form, the relations for the covering surface of S will automatically have this property. It therefore follows that a covering surface of S is orientable if S is orientable (see Definition 4.1). The converse of this is not true (see Theorem 6.1).

Covering surfaces of a sphere are called Riemann surfaces [see Knopp, pages 93–118, 139–142]. Riemann, one of the founders of modern topology, studied such surfaces because they served to uniformize multiple-valued functions of a complex variable (see Examples 1.2, 5.2, 5.3). An excerpt from Riemann's fundamental paper is given in the *Source Book in Mathematics* (Smith, pages 404-410). The references made to the theory of complex variables in the following examples are made for the benefit of the readers who have some knowledge of this field. They may be ignored by one who is interested only in the covering surfaces themselves.

Example 5.1. The relation $aa^{-1}bb^{-1}cc^{-1} = 1$ describes a sphere, as shown in Figure 41. Each of the four relations $a_i a_i^{-1}\, b_i b_i^{-1}\, c_i c_i^{-1} = 1$

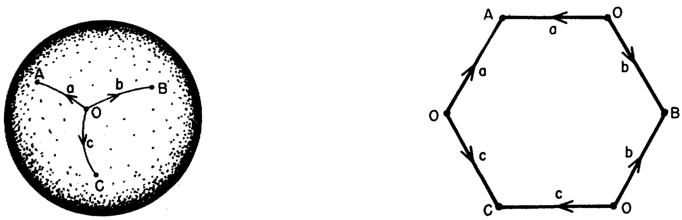


Fig. 41

(i = 1,2,3,4) also represents a sphere. The following four relations are obtained by permuting the subscripts on the second occurrences of each letter.

$$\begin{cases} a_1 a_4^{-1} \quad b_1 b_1^{-1} \quad c_1 c_4^{-1} = 1, \\ a_2 a_1^{-1} \quad b_2 b_3^{-1} \quad c_2 c_1^{-1} = 1, \\ a_3 a_2^{-1} \quad b_3 b_4^{-1} \quad c_3 c_2^{-1} = 1, \\ a_4 a_3^{-1} \quad b_4 b_2^{-1} \quad c_4 c_3^{-1} = 1. \end{cases}$$

These relations define a 4-sheeted covering surface of the sphere of Figure 41. This covering surface is represented by the diagram of Figure 42. In this figure, the convention is used that whenever a symbol without a subscript is written on a face, it is to be given the number of the face as a subscript. Each point of the sphere

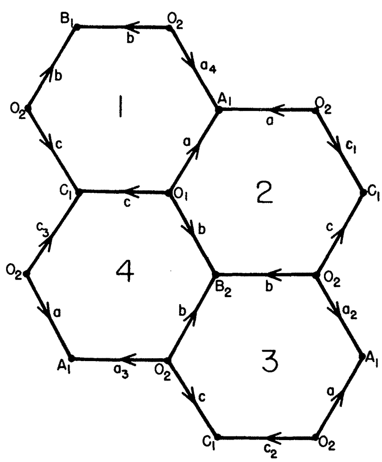


Fig. 42

that is not on one of the edges a, b, c is covered four times - once by each of the similarly located points on the four sheets. The same is true for each point which is on only one of the edges a, b, c since each edge has four copies.

But no one of the vertices A, B, C, O is covered by four similarly located points. For example, the vertex O is the left vertex of each of the symbols a, b, c (Figure 41). Any vertex of the covering surface which is the initial vertex of any of the symbols a_i, b_i, c_i (i = 1,2,3,4) is in a similar position and is said to cover O. If the initial vertex of a_1 is called O_1, then successive use of the second, fourth, and first of the above relations shows that O_1 is also the initial vertex of b_2, the initial vertex of c_4, and again the initial vertex of a_1. Thus O_1 is a vertex in the covering surface and is the initial point for each of the edges in the *circular system* $a_1 b_2 c_4$ which surrounds O_1 in Figure 42. The point O_2 has the circular system $(a_4 b_1 c_1)(a_2 b_3 c_2)(a_3 b_4 c_3)$.

Figure 43 shows the circular system of edges and faces around O_2. The point O_2 is a *branch point* of order 2, because three sheets of the surface are joined at O_2. This is represented symbolically by the fact that the product abc occurs three times in the circular system of edges for O_2. In general, the order of a branch point is one less than the number of sheets joined at the point (see Definition 5.1 below). The remaining vertices and their systems of edges are A_1: $a_2^{-1}\ a_1^{-1}\ a_4^{-1}\ a_3^{-1}$; B_1: b_1^{-1} ; B_2: $b_2^{-1}\ b_3^{-1}\ b_4^{-1}$; C_1: $c_1^{-1}\ c_4^{-1}\ c_3^{-1}\ c_2^{-1}$. Vertices O_2, A_1, B_2, C_1 are branch points of orders 2,3,2,3, respectively. The total of these orders is 10 and is the number of vertices "missing" in the sense discussed below.

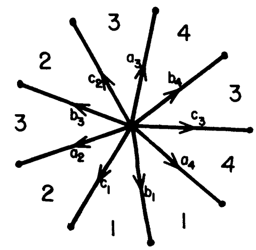


Fig. 43

To determine the Euler characteristic χ of this covering surface, note that the system of relations has four times as many symbols and four times as many relations as the surface itself. If there had been four times as many vertices, χ would also have been four times as large (see Definition 4.5). Since $\chi = 2$ for the sphere and 10 vertices are "missing", $\chi = 4\cdot 2 - 10 = -2$ for the covering surface. Since this surface is closed and orientable, $\chi = 2 - 2p$ and $q = r = 0$. Therefore $p = 2$ and the surface is combinatorially equivalent to a sphere with two handles.

If the process of determining vertices for a surface is analyzed, it can be seen that a vertex which has a circular system of edges $(ab\ldots p)$ on the original surface will be "covered" by certain vertices whose circular system is of the type $(ab\ldots p)(ab\ldots p)\ldots(ab\ldots p)$, with subscripts on each symbol. If the surface is n-sheeted, each of the symbols $a, b, \ldots, p$ has n copies and there are therefore n groupings of type $(ab\ldots p)$ among the circular systems for the covering points. Because of these facts, the number b of the following definition is a non-negative integer. Also, the sum of the orders of the branch points covering a vertex P is equal to the difference between n and the number of covering points of P. Then the number of vertices on an n-sheeted covering surface is equal to n times the number of vertices on the surface itself less the sum of the orders of the branch points. The sum of the orders of the branch points is in this sense the number of vertices "missing" in the covering surface.

Definition 5.1. Let P be a vertex on a surface S and let $\bar{S}$ be an n-sheeted covering surface of S. If P is the initial [terminal] vertex of a symbol x in S, then the initial [terminal] vertices of $x_1, \ldots, x_n$ in $\bar{S}$ are *covering points* of P. Let $\bar{P}$ be a covering point of P and let $b + 1$ be the number of symbols of which $\bar{P}$ is a vertex divided by the number of symbols of which P is a vertex. If $b \geq 1$, then $\bar{P}$ is said to be a *branch point of order b*.

As noted before, an n-sheeted covering surface of a sphere is connected and closed ($q = r = 0$). There are n times as many symbols and n times as many relations as for the sphere. All vertices are covering points of vertices for the sphere and the number of vertices on the covering surface is $nV - \beta$, where V is the number of vertices on the sphere and β is the sum of the orders of the branch points. Thus

$$\chi = 2n - \beta.$$

Since $\chi = 2 - 2p$, the value of χ completely determines the topological nature of a covering surface of a sphere. These methods will be used in analyzing the following examples. Each of the Examples 5.2 and 5.3 verify the truth of the following theorem (also see Theorem 6.1).

Theorem 5.1. *Any closed orientable surface can be represented as a 2-sheeted Riemann surface.*

Example 5.2. Consider the 2-sheeted Riemann surface which consists of two sheets covering the sphere of Figure 44, with these sheets crossing each other along the paths $b^1, b^2, \dots, b^m$ (the case $m = 2$ was introduced as Example 1.2). If this surface is cut along the closed path $a^1b^1a^2b^2\dots a^mb^m$, each covering sheet of the sphere will be cut into two pieces. The two pieces of the outer [inner] sheet were joined along each a^i, and each hemisphere of the outer sheet was joined to the other hemisphere of the inner sheet along each $b^1, b^2, \dots, b^m$. The relations for the sphere of Figure 44 are:

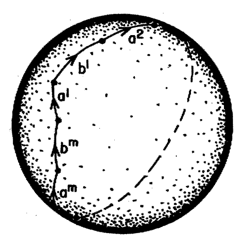


Fig. 44

$$a^1\, b^1\, a^2\, b^2 \dots a^m b^m = 1; \qquad a^1\, b^1\, a^2 b^2 \dots a^m b^m = 1.$$

If two copies of this sphere are made by introducing subscripts 1 for one copy and subscripts 2 for the other, then the relations for the covering surface are obtained by interchanging (for each k) the second occurrence of b_1^k and the second occurrence of b_2^k in the same way b_1 and b_2, and d_1 and d_2, were interchanged in Example 1.2. These four relations for the covering surface are:

$$a_1^1\, b_1^1\, a_1^2\, b_1^2 \dots a_1^m\, b_1^m = 1, \quad a_1^1\, b_2^1\, a_1^2\, b_2^2 \dots a_1^m\, b_2^m = 1,$$

$$a_2^1\, b_2^1\, a_2^2\, b_2^2 \dots a_2^m\, b_2^m = 1, \quad a_2^1\, b_1^1\, a_2^2\, b_1^2 \dots a_2^m\, b_1^m = 1.$$

For each k, the terminal vertex of a^k has a circular system of edges $(a^k)^{-1}b^k$. For the covering surface, the terminal vertex of a_1^k has the circular system of edges $[(a_1^k)^{-1}b_1^k]\,[(a_2^k)^{-1}b_2^k]$ and the initial vertex of a_1^k has the circular system $[a_1^k(b_2^{k-1})^{-1}]\,[a_2^k(b_1^{k-1})^{-1}]$, where k-1 is replaced by m if $k = 1$. Thus each vertex is covered by a single point and each vertex of the covering surface is a branch point of order 1. The sum of the orders of the branch points is $\beta = 2m$, and therefore $\chi = 4 - 2m = 2 - 2p$. Therefore the covering surface is of genus $p = m - 1$. It is a sphere if $m = 1$ and a torus if $m = 2$.

The Riemann surface of Example 5.2 uniformizes the function

$$w^2 = [1 - (k_1 z)^2]\,[1 - (k_2 z)^2]\ldots[1 - (k_m z)^2],$$

where $k_1, \ldots, k_m$ are distinct complex numbers. This surface can be formed as a 2-sheeted covering of the complex plane which crosses itself along cuts $b^1, b^2, \ldots, b^m$, where the path $a^1b^1a^2b^2\ldots a^mb^m$ is a closed polygonal path which joins the points $\pm 1/k_1, \ldots, \pm 1/k_m$ in some order (but does not cross itself). By stereographic projection, this surface becomes the 2-sheeted covering surface of the sphere shown in Figure 44.

Example 5.3. Consider the 2-sheeted Riemann surface which consists of two sheets covering the sphere of Figure 45, with the sheets crossing each other along the paths $a^1, a^2, \ldots, a^n$. The sphere has the relation:

$$a^1(a^1)^{-1}a^2(a^2)^{-1}\ldots a^n(a^n)^{-1} = 1.$$

The covering surface has the relations:

$$\begin{cases} a_1^1(a_2^1)^{-1}a_1^2(a_2^2)^{-1}\ldots a_1^n(a_2^n)^{-1} = 1, \\ a_2^1(a_1^1)^{-1}a_2^2(a_1^2)^{-1}\ldots a_2^n(a_1^n)^{-1} = 1. \end{cases}$$

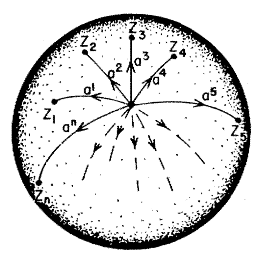


Fig. 45

If n is odd, the vertex, O, which is the initial vertex of each a^i, has a covering point O_1 which has the circular system of edges $[a_1^1a_2^2a_1^3a_2^4\ldots a_1^n]\,[a_2^1a_1^2\ldots a_2^n]$ and is a branch point of order 1. If n is even, O is covered by the two points O_1 and O_2, where O_1 has the circular system of edges $[a_1^1a_2^2\ldots a_2^n]$ and O_2, has the circular system of edges $[a_2^1a_1^2a_2^3\ldots a_1^n]$ (in this case O_1 and O_2 are not branch points). Each of the other vertices of the covering surface is a branch point of order 1, for Z_k is covered by only one point, with the circular system of edges $[(a_1^k)^{-1}]\,[(a_2^k)^{-1}]$. Therefore the sum of the orders of the branch points is $\beta = n + 1$ if n is odd and $\beta = n$ if n is even. But $\chi = 4 - \beta = 2 - 2p$, so the surface is of genus $p = (n-1)/2$ if n is odd and $p = (n-2)/2 = m-1$ if $n = 2m$. If $n = 1$ or $n = 2$, the Riemann surface is a sphere. For $n = 3$ or $n = 4$, the surface is a torus.

The Riemann surface of Example 5.3 uniformizes the function

$$w^2 = (z - z_1)(z - z_2) \ \ldots \ (z - z_n),$$

where $z_1, \ldots, z_n$ are distinct complex numbers [see Knopp, pages 112–118]. This surface can be formed as a 2-sheeted covering of the complex plane, which crosses itself along cuts a^i from ∞ to z_i. By stereographic projection with $z = \infty$ at O, this surface becomes the 2-sheeted covering of the sphere shown in Figure 45.

Example 5.4. In 1809 Poinsot described two polyhedra, called the *great dodecahedron* and the *great icosahedron,* which have properties of regularity like the five classical regular polyhedra [see Ball and Coxeter, pages 143–145, and Figures 34, 30 opposite page 134].

The great dodecahedron may be obtained from a regular icosahedron (Figure 46) by deleting its 20 triangular faces and inserting 12 pentagonal faces to fit the 12 regular pentagonal perimeters (such as $\bar{A}D\bar{E}\bar{C}\bar{F}$). It can be thought of as a Riemann surface [the sphere is defined by 20 relations (one for each face of the icosahedron) and the Riemann surface is based on these relations by radial projection onto the icosahedron; e.g. the pentagonal face $\bar{A}D\bar{E}\bar{C}\bar{F}$ is represented by 5 relations corresponding to 5 triangles having a common vertex below (but covering) $\bar{B}$]. The surface is 3-sheeted and has 12 branch points

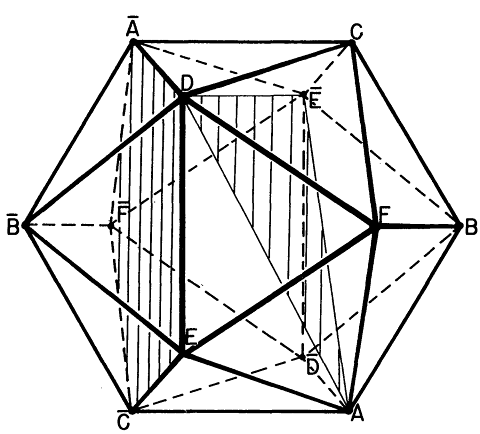


Fig. 46

of order 1, since two sheets of the great dodecahedron are joined at each vertex of the icosahedron. Thus the sum β of the orders of the branch points is 12 and $\chi = 3 \cdot 2 - 12 = -6 = 2 - 2p$. Therefore $p = 4$ (using the formula $\chi = V - E + F$ gives $\chi = 12 - 30 + 12 = -6$). The great dodecahedron is therefore a closed orientable surface of genus 4 (a sphere with 4 handles).

The great icosahedron may be obtained from a regular icosahedron by joining each vertex by a line segment (edge) to each of the other five vertices which are neither adjacent nor antipodal to it and then inserting 20 triangular faces to fit the 20 equilateral triangular perimeters (such as $D\bar{E}\bar{D}$) to be found in the network formed by the 30 new edges and the 12 original vertices (the original faces and edges are deleted). This is also a Riemann surface. It is 7-sheeted with 12 branch points of order 1, so $\chi = 7 \cdot 2 - 12 = 2$ and $p = 0$ (using $\chi = V - E + F$ gives $\chi = 12 - 30 + 20 = 2$). The great icosahedron is therefore combinatorially equivalent to a sphere.

As shown before, the Euler characteristic of an n-sheeted covering surface of a sphere is $\chi = 2 \cdot n - \beta$, where β is the sum of the orders of the branch points. Since $\chi \leq 2$ for any surface, it follows that $\beta = 0$ only if $n = 1$. Thus any non-trivial covering of the sphere has branch points. Such a covering surface is closed and orientable, so χ is even and the value of β is at least $2n - 2$.

6. *Unbranched Coverings.*

As described above, a covering can be formed for any surface for which some symbol occurs twice. This is done by making copies and permuting in one occurrence of one or more symbols in such a way that the sheets are connected. It has been seen that a covering surface is orientable (or closed) if the covered surface is orientable (or closed) and that a covering surface of the sphere has branch points if it has more than one sheet. The two classes of coverings which have been of most interest are Riemann surfaces and unbranched coverings. Because of the possibility of reducing a surface to canonical form, it will be possible to establish some general results about unbranched coverings by means of examples. The whole process of formimg and analyzing a covering surface can be carried out in a completely abstract symbolic fashion, but if the permutation is done at random the covering is likely to have branch points. Unbranched coverings are the result of a special form of weaving and are usually gotten with a specific geometric scheme in mind, as illustrated by Examples 6.4 and 6.5 [also see Seifert and Threlfall, Chapter 8].

Example 6.1. The projective plane can be thought of as a disk with diametrically opposite points identified. It can be described by the relation $cc = 1$. The two copies $c_1 c_1 = 1$, $c_2 c_2 = 1$ can be woven together in the following way (and only this way): $c_1 c_2 = 1$, $c_2 c_1 = 1$. Thus the sphere is an orientable "double" of the projective plane (a 2-sheeted covering without branch points).

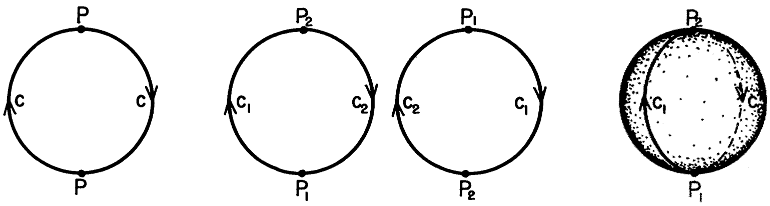


Fig. 47

Example 6.2. The Klein bottle is combinatorially equivalent to a sphere with two crosscaps. It can be described by the relation $aba^{-1}b = 1$. Figure 48 shows the 2-sheeted unbranched covering defined by the relations given below. This "double" of the Klein bottle is a torus.

$$\begin{cases} a_1 b_1 a_1^{-1} \; b_2 = 1, \\ a_2 b_2 a_2^{-1} \; b_1 = 1. \end{cases}$$

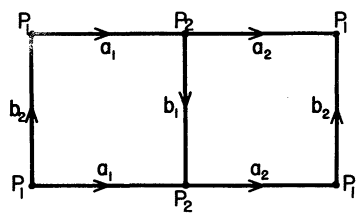


Fig. 48

It follows from Examples 6.1 and 6.2 that any nonorientable surface has a 2-sheeted orientable unbranched covering (this is also one of the conclusions which follows from Example 6.3). For in the case of a sphere with an odd number of crosscaps, all but one of the crosscaps can be changed into handles (the inverse of Step 5 of the reduction to canonical form). Two copies of the surface can then be woven together as in Example 6.1. If the sphere has an even number of crosscaps, all but two crosscaps can be changed into handles and the remaining two form a Klein bottle. Two copies of the surface can then be woven together as in Example 6.2.

Example 6.3. Four closed surfaces are defined by the following four sets of relations:

$$P: \quad ccxx^{-1} = 1. \qquad S: \quad S: \quad c_1c_2x_1x_1^{-1} = 1, \quad c_2c_1x_2x_2^{-1} = 1.$$

$$C_n: \begin{cases} c^1c^1x^1(x^2)^{-1} = 1, \\ c^2c^2x^2(x^3)^{-1} = 1, \\ c^3c^3x^3(x^4)^{-1} = 1; \\ \vdots \\ c^nc^nx^n(x^1)^{-1} = 1. \end{cases} \quad H_{n-1}: \begin{cases} c_1^1c_2^1x_1^1(x_1^2)^{-1} = 1, & c_2^1c_1^1x_2^1(x_2^2)^{-1} = 1, \\ c_1^2c_2^2x_1^2(x_1^3)^{-1} = 1, & c_2^2c_1^2x_2^2(x_2^3)^{-1} = 1, \\ c_1^3c_2^3x_1^3(x_1^4)^{-1} = 1, & c_2^3c_1^3x_2^3(x_2^4)^{-1} = 1, \\ \vdots & \vdots \\ c_1^nc_2^nx_1^n(x_1^1)^{-1} = 1, & c_2^nc_1^nx_2^n(x_2^1)^{-1} = 1. \end{cases}$$

The surface P is the projective plane and S is a sphere. Since the block c^1c^1 occurs in the relations for C_n, the surface C_n is nonorientable. The surface H_{n-1} is orientable, since by inverting the relations in the right column each symbol can be made to occur once inverted and once not inverted (see Definition 4.1). As indicated by the notation, S is a 2-sheeted covering of P; C_n and H_{n-1} are n and $2n$-sheeted coverings of P; H_{n-1} is an n-sheeted covering of S and a 2-sheeted covering of C_n. The two vertices of P have the circular systems of edges $c^{-1}xc$ and x^{-1}. Each is covered twice by vertices of S. Thus S is an unbranched covering of P. Also:

(1$'$). The surface C_n has two vertices, with the circular systems of edges $[(c^1)^{-1}x^1c^n][(c^n)^{-1}x^n c^{n-1}] \ldots [(c^2)^{-1}x^2c^1]$ and $[(x^1)^{-1}]$ $[(x^2)^{-1}] \ldots [(x^n)^{-1}]$. Thus C_n is an n-sheeted covering of P with two branch points of order $n-1$. Then $\chi = n \cdot 1 - 2(n-1) = 2-n$, since $\chi = 1$ for P. Hence C_n is a closed nonorientable surface with n crosscaps.

(1$''$). The surface H_{n-1} has four vertices, with the circular systems of edges: $[(c_2^1)^{-1}x_1^1c_1^n][(c_2^n)^{-1}x_1^nc_1^{n-1}] \ldots [(c_2^2)^{-1}x_1^2c_1^1]$, $[(c_1^1)^{-1}x_2^1c_2^n]$ $[(c_1^n)^{-1}x_2^nc_2^{n-1}] \ldots [(c_1^2)^{-1}x_2^2c_2^1]$, $[(x_1^1)^{-1}][(x_1^2)^{-1}] \ldots [(x_1^n)^{-1}]$, and $[(x_2^1)^{-1}][(x_2^2)^{-1}] \ldots [(x_2^n)^{-1}]$. Thus H_{n-1} is a $2n$-sheeted covering of P with four branch points, each of order $n-1$. Hence $\chi = (2n) \cdot 1 - 4(n-1)$ $= 2-2(n-1)$ and H_{n-1} is a closed orientable surface with n-1 handles.

(2) Since each vertex of C_n is covered twice by vertices of H_{n-1}, H_{n-1} is a 2-sheeted covering of C_n with no branch points.

(3) Since each vertex of S is covered only once by vertices of H_{n-1}, H_{n-1} is an n-sheeted covering of S with four branch points of order $n-1$.

These facts can be summarized as the following theorem.

Theorem 6.1. *(1) Any closed surface (orientable or not) can be represented as a covering surface of the projective plane.*

(2) Any closed nonorientable surface has a 2-sheeted unbranched covering which is closed and orientable.

(3) Any closed orientable surface can be represented as a Riemann surface with four branch points of equal order.

Example 6.4. The relation $aba^{-1}b^{-1}cc = 1$ defines a sphere with one handle and one crosscap, which is also a torus with one crosscap (Figure 1 without the cuff). Several examples of covering surfaces will be developed simultaneously by use of the following sets of relations. The notation indicates that S_2, S_3, and S_6 are 2, 3, and 6-sheeted coverings of S_1, and that S_6 is a 3-sheeted covering of S_2 and a 2-sheeted covering of S_3.

$$S_1:\ a\,b\,a^{-1}\,b^{-1}\,c\,c=1.\quad S_2:\ a^1b^1(a^1)^{-1}(b^1)^{-1}c^1c^2=1.\ \ a^2b^2(a^2)^{-1}(b^2)^{-1}c^2c^1=1,$$

$$S_3:\begin{cases} a_1b_1a_1^{-1}\,b_2^{-1}\,c_1c_1=1,\\ a_2b_2a_2^{-1}\,b_3^{-1}\,c_2c_2=1,\\ a_3b_3a_3^{-1}\,b_1^{-1}\,c_3c_3=1.\end{cases} S_6:\begin{cases} a_1^1b_1^1(a_1^1)^{-1}(b_2^1)^{-1}c_1^1c_1^2=1,\ \ a_1^2b_1^2(a_1^2)^{-1}(b_2^2)^{-1}c_1^2c_1^1=1,\\ a_2^1b_2^1(a_2^1)^{-1}(b_3^1)^{-1}c_2^1c_2^2=1,\ \ a_2^2b_2^2(a_2^2)^{-1}(b_3^2)^{-1}c_2^2c_2^1=1,\\ a_3^1b_3^1(a_3^1)^{-1}(b_1^1)^{-1}c_3^1c_3^2=1,\ \ a_3^2b_3^2(a_3^2)^{-1}(b_1^2)^{-1}c_3^2c_3^1=1,\end{cases}$$

It can be easily verified that the surfaces S_1, S_2, S_3, and S_6 have 1, 2, 3, and 6 vertices, respectively. Therefore no one of the covering surfaces has branch points. The Euler characteristic of an n-sheeted unbranched covering surface is equal to $n \cdot \chi$, where χ is the Euler characteristic of the base surface. Since $\chi = -1$ for S_1, the value of χ for S_2, S_3, and S_6 are -2, -3, and -6. Of course, the number of symbols (edges or segments) and the number of relations (faces or regions) in these four surfaces are also proportional to 1, 2, 3, 6. All four surfaces are closed; S_1 and S_3 are nonorientable, since the blocks cc and c_1c_1 occur in their relations; S_2 and S_6 are orientable, since by inverting the right-hand relations each symbol can be made to appear once inverted and once not inverted (see Definition 4.1).

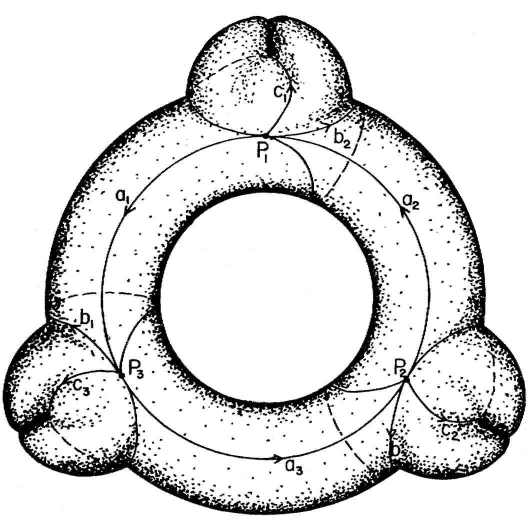


Fig. 49

For surface S_3, $\chi = -3 = 2 - 2p - q$. Thus S_3 can be interpreted as a torus ($p = 1$) with 3 crosscaps (or as a sphere with 5 crosscaps). With this interpretation, it can be represented as in Figure 49. The cuts shown divide the surface into the three pieces described by the above relations. It can be seen to be a 3-sheeted unbranched covering of a torus with one crosscap by imagining that it is cut around a circle between two crosscaps and rejoined along the cut after being pulled through itself for two revolutions so that the three crosscaps are superimposed.

The surface S_6 has $\chi = -6 = 2 - 2p$ and is a sphere with four handles. It is represented in Figure 50 with the cuts which produce the pieces described by the above relations. It can be seen to be a 3-sheeted unbranched covering of a torus with one handle (S_2) by the same method of pulling it through itself as used for Figure 49.

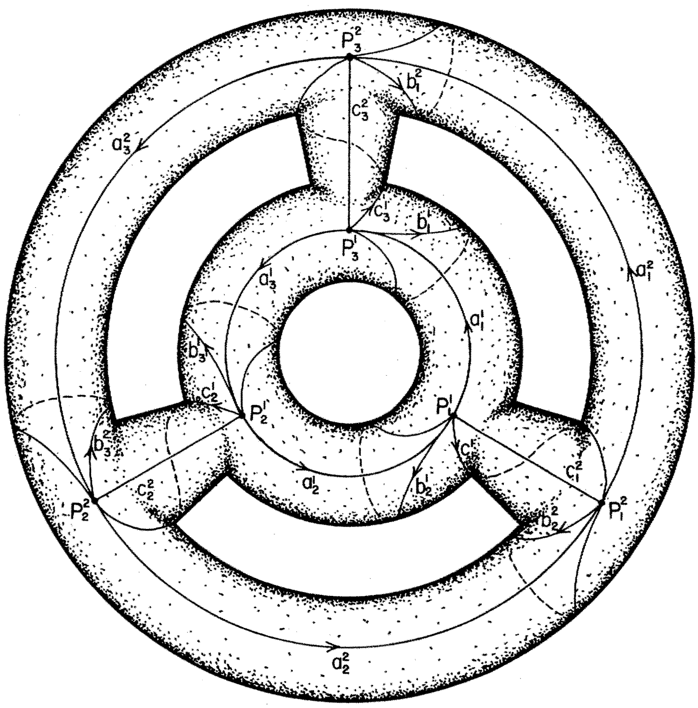


Fig. 50

Example 6.5. The relation $aba^{-1}b = 1$ defines a Klein bottle. The following sets of relations are based on this dissection of the Klein bottle. The notation indicates theat R_2, R_3, and R_6 are 2, 3, and 6-sheeted covering surfaces of R_1, and that R_6 is a 3-sheeted covering of R_2 and a 2-sheeted covering of R_3.

$$R_1: \quad a\,b\,a^{-1}b = 1. \qquad R_2: \quad a^1b^1(a^1)^{-1}b^2 = 1, \qquad a^2b^2(a^2)^{-1}b^1 = 1.$$

$$R_3: \begin{cases} a_1b_1a_2^{-1}b_3 = 1, \\ a_2b_2a_3^{-1}b_2 = 1, \\ a_3b_3a_1^{-1}b_1 = 1, \end{cases} \qquad R_6: \begin{cases} a_1^1b_1^1(a_2^1)^{-1}b_3^2 = 1, & a_1^2b_1^2(a_2^2)^{-1}b_3^1 = 1, \\ a_2^1b_2^1(a_3^1)^{-1}b_2^2 = 1, & a_2^2b_2^2(a_3^2)^{-1}b_2^1 = 1, \\ a_3^1b_3^1(a_1^1)^{-1}b_1^2 = 1, & a_3^2b_3^2(a_1^2)^{-1}b_1^1 = 1. \end{cases}$$

It can be easily verified that the surfaces R_1, R_2, R_3, and R_6 have 1, 2, 3, and 6 vertices, respectively. Therefore no one of the covering surfaces has branch points. Since $\chi = 0$ for R_1, it follows that $\chi = 0$ for all four surfaces. If the right hand relations for R_2 and R_6 are inverted, then no symbol occurs twice in the same form. Hence R_2 and R_6 are closed orientable surfaces with $\chi = 2 - 2p = 0$. Thus $p = 1$ and R_2 and R_6 are each combinatorially equivalent to a torus. In the relations for R_3, each a_i occurs once as a_i and once as a_i^{-1}, but no b_i can be made to have this property without destroying it for one of the a_i's. Hence R_3 is nonorientable (Definition 4.1) and has $\chi = 2 - q = 0$.

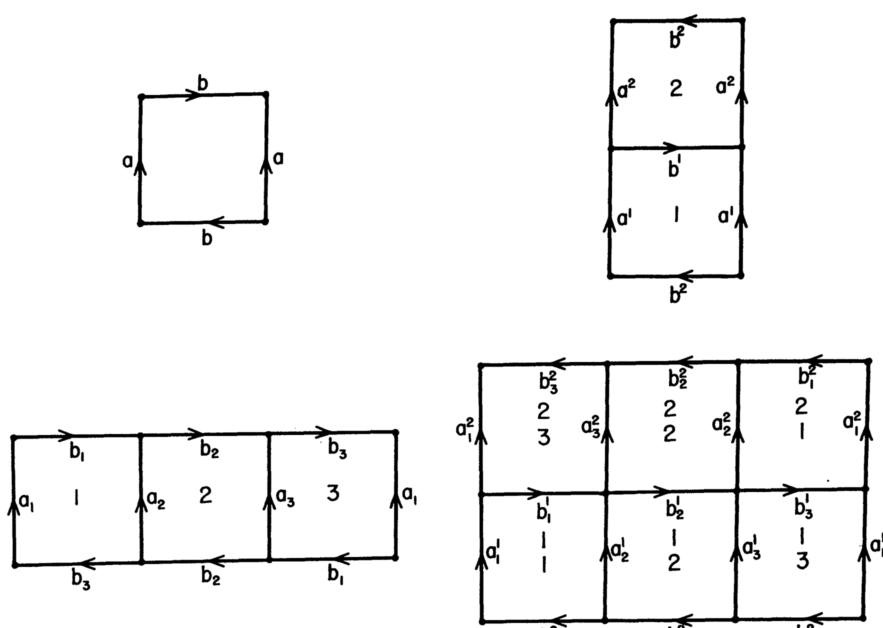


Fig. 51

Thus $q = 2$, and R_1 and R_3 are each combinatorially equivalent to a Klein bottle. Thus we have a Klein bottle (R_3) which is a "triple" of a Klein bottle (R_1), a torus (R_2 or R_6) which is a "double" of a Klein bottle (R_1 or R_3) and a torus (R_6) which is a "triple" of a torus. These coverings can be represented as in Figure 51.

REFERENCES

Ball, W.W.R., *Mathematical Recreations and Essays,* 11th ed., revised by H.S.M. Coxeter, Macmillan, Chap. V.

Brahana, H.R., "Systems of Circuits on Two-Dimensional Manifolds," *Annals of Mathematics,* vol. 23 (1921-23), pages 144-168.

Courant, R., and Robbins, H., *What is Mathematics?,* Oxford University Press, Chapter V.

Fréchet, M., and Ky Fan, *Introduction à la Topologie Combinatoire (1),* Paris, Librairie Vuibert, 1946.

Hilbert, D., and Cohn-Vossen, S., *Geometry and the Imagination,* Chelsea, 1952, Chapter VI.

Knopp, K., *Theory of Functions (Part Two),* Dover Publications, 1947, Chapters 4 and 6.

Lefschetz, S., *Introduction to Topology,* Princeton University Press, 1949.

Newman, M.H.A., *Elements of the Topology of Plane Sets of Points,* Cambridge University Press, Second Edition (reprinted), 1954.

Seifert, H., and Threlfall, W., *Lehrbuch der Topologie,* Teubner-Chelsea, 1947.

Smith, D.E., *A Source Book in Mathematics,* McGraw-Hill, 1929, pages 404-410.

Tucker, A.W., and Bailey, H.S., Jr., "Topology," *Scientific American,* January 1950, pages 18-24.

Wilder, R., "Topology," *The Encyclopaedia Britannica,* XXII, 298-298F (in recent reprinting *only*).

Haverford College.

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

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In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

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COMMENT ON L. CANERS' "PYTHAGOREAN PRINCIPLE AND CALCULUS" *

H. W. Becker

The petitio principii is that in taking $D_\phi(uz^{-2}) = 0$, the author has assumed uz^{-2} constant, in advance of proving it.

This ingenious fallacy calls attention to Prof. Elisha Loomis' book "The Pythagorean Proposition" (Edwards Bros., Ann Arbor, 1940). He compiles 353 proofs of the Pythagorean theorem, by algebra, geometry, quaternions, and vector analysis. On p. 244, he asserts that no independent proof can be based on trig., analytic geometry, or calculus, since they are themselves based on the Pythagorean theorem. Is he absolutely right, at the sophomore level? At the graduate level?

A plurality of the 353 proofs is due to Prof. Loomis himself. Many are credited to bright high school students. One bears the name of President James A. Garfield, who worked it out as a U.S. Senator, while discussing the subject with his colleagues in the Senate corridors. What math. topic today, has the vitality to engross the coversation of U.S.Senators? (What senator has the vitality to converse about Math.? Ed.)

The book was sponsored by an Ohio Masonic organization. Why? In Mackey's "Encyclopedia of Masonry," we find "Right Angle. A symbol of upright conduct." Properly interpreted, the Pythagorean theorem appears an answer to juvenile delinquency, the crime wave, and the temptations of Public office.

(continued on Pg.58)

* Mathematics Magazine, 28 (1955) p. 276.

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, as *a teacher*, are intetested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

GEOMETRY IN GENERAL EDUCATION

R. E. Horton

It is not possible to give a precise definition of General Education which would be acceptable to most educators. However certain elements are rather generally agreed upon. General Education is education for the total personality and the whole realm of living rather than for the specifics of earning a living. As such it attempts to develop certain desirable attitudes, provide certain knowledge and increase certain skills.

The area of attitudes might include such items as a positive and constructive acceptance of civic responsibility, an unprejudiced tolerance of differences of race, creed and social customs, the openminded approach to new ideas and a constructive attitude toward the use of leisure time.

The area of knowledge usually found in General Education programs includes a broader understanding of our cultural heritage, our individual selves and our physical environment. The part of General Fducation devoted to skills deals principally with communication skills, methods of employing reasoning and skill in calculation or computation. It is the purpose of this paper to discuss the place of geometry in a program of General Education and to suggest a course which would adequately serve to achieve some of the goals of General Education.

An examination of the history of higher education reveals that the Geometry of Euclid has been a prerequisite to collegiate matriculation or advancement since very early days. This requirement probably endured the test of time because it generally was felt that geometry filled three needs. It was necessary for further scientific study. It screened out those who were considered unfit for higher education. And it developed a way of reasoning. This last characteristic was probably the strongest reason why geometry was required even for scholars in non-scientific fields of study.

Recent years have seen many attacks made upon the requirement of geometry for non-science majors. The value of geometry as a study which produced general reasoning ability was questioned by psychologists who discounted the whole theory of the transfer of learning. Educators used the argument of democracy in education to invalidate the use of geometry as a screening device for non-science majors. As a result of these pressures those who felt that geometry should continue to be required for all students had to re-examine their position. This has resulted in various changes. In particular it has led to the development of new courses in geometry which would achieve the needs of non-science majors relative to General Education. The question of non transfer of learning has been effectively solved by careful relating of abstract material to experiences in the daily life of the student.

One such experimental course recently developed at Los Angeles City College will be discussed here. An examination of the General Education needs of non-science majors led to the belief that a properly designed course in geometry could make unique contributions toward three objectives. Geometry could be taught so as to develop an effective way of reasoning. Geometry can reveal a certain body of knowledge about our physical environment not to be found in other courses. Finally, geometry can be taught in a way that will provide students with an awareness of the contributions of mathematics and mathematicians to our culture and civilization. The course was designed to meet the geometric requirements of all students who were not studying geometry as a prerequisite for further mathematics study.

The curriculum is based upon a test book in Plane Geometry, a body of supplementary classroom material and extensive collateral reading. The part of the curriculum devoted to the development of effective reasoning contains a number of departures from the traditional geometry course. At the outset a dictionary is examined and word definitions are analyzed for examples of circular reasoning. This leads to an understanding of the need for a few basic undefined elements. Next the postulates of geometry are studied and some of the consequences of changing our postulates are considered. At this time students are required to examine in search of postulates such other areas of thinking as politics, sociology, religion and advertizing.

The rules of logic and their illustration by Euler diagrams are studied. Simple syllogisms are analized along with geometric theorems. Simple theorems are proved and analogous statements found in other life situations are examined to determine whether they can be proven by similar methods. This leads to a study of the meaning of Proof and to the general methods of reasoning. Direct and indirect methods of proof are applied to both geometric and non-geometric problems. This importance of valid reasons for all statements made is emphasized.

Careful attention is given to include curriculum material which would develop knowledge about our physical environment. The basic figures and objects of geometry are related to household articles and other objects found about us in our daily life. The geometry of architecture, home furnishings and art are topics which appeal to many non-science majors. The essential difference between the process of counting and of measuring is discussed. This leads to a study of the geometric nature of space, distance, angle and other measured concepts. The properties of geometric figures are then studied as examples of laws of nature which can be discovered by inductive and deductive reasoning processes.

The cultural contributions of geometry are related to the great ideas of our civilization. The geometry of the Greeks is discussed in connection with Greek philosophy and art. The renaissance of mathematics is related to the renaissance of art, literature and philosophy and the reformation in religion. The necessity for non-Euclidean geometries before the modern discoveries of science could be made is discussed. Rigorous proofs and precise definitions are not necessary here as this part of the course is trying to develop an appreciation rather than a working knowledge of the subject. Collateral reading, student reports and term papers are used to carry out this part of the curriculum.

The broader objectives of the course and the different emphasis of the curriculum material requires alterations in the methods of teaching used in more traditional geometry classes. Less emphasis is placed upon the purely mathematical approach. This does not mean that no theorems are proven. Rather, the theorems which are proved are examined along with analagous material from other subjects from the point of view of methods of proof rather than just considering the mathematical results.

Wide student participation is gained through oral reports and term papers. Students are required to search in papers and magazines for analogues to geometric problems. Advertizing samples are analyzed for implicit hypotheses and logical consistency.

The course described herein is the result of an experimental geometry course presented for the last three semesters at Los Angeles City College. Changes in curriculum and methods have been and are being made. However, the broad framework of the course is fairly well defined as described above. The results in the experimental classes have validated the belief that the three objectives can be achieved. Subjective judgments of the instructors involved agree that the new course is superior to the traditional course for non-science students in every important aspect.

BIBLIOGRAPHY

1. Diekhoff, John S., *Democracy's College*, Harper Bros., 1950. Chapter IV 378. 15 D 563d

2. Hudson, Hoyt H., *Educating Liberally*, Stanford University Press, 1945. Chapter IV, 378 H 868e

3. Johnson, B. Lamar, *General Education In Action,* American Council on Education, 1952. 378.73 C 1282g
4. Northrop, Eugene P., *Fundamental Mathematics,* University of Chicago Press, 1944. Vol.1, Chapter 1-7.
5. Nathional Conference on Higher Education Report, *Current Trends in Higher Education,* N.EAA., 1948.

   Part III- Peacock, M., *Coordination of Professional and General Education*
   Trowbridge, Hoyt, *General Education in Liberal Arts Colleges.*
   Lovinger, W.C., *General Education for the Prospective Teacher.*
6. Sturm, Harold E., *Mathematics for Non-Science Majors,* School Science and Mathematics, Vol. L, Number 6, June 1950.
7. Urner, Samuel E., *Unified Freshman Mathematics,* Mathematics Magazine, Vol. 25, Number 1, September 1951.
8. Woodby, Lauren G., *Junior College Course in Mathematics for General Education,* School Science and Mathematics, Vol.LIII, No. 9, December 1953
9. Wynne, John P., *General Education in Theory and Practice,* Brooklyn Associates 1952, 370.1 W 992g
10. Zant, James H., *General Mathematics at the College Level,* School Science and Mathematics, Vol. L, Number 6, June 1950.

Los Angeles City College.

Errata in

"Tchebycheff Inequalities as a Basis for Statistical Tests"

C. D. Smith - Vol. 28 No. 4.

Page	Line	Corrections
188	13	$P_{\lambda\sigma} \leq$
188	20	$\phi = \dfrac{[2cn/\lambda(2n+1)]^{2n}}{(2n+1)(\lambda/c - 1)}$
189	4	Change P_i to p_i in the equation $P_d = \sum_{i=c+1}^{n} P_i$
190	39	Change "propossl" to proposal.
191	8	After "... small samples." The sentence omitted is, R.A.Fisher developed student's $F(Z)$ in the form $F_n(t) = \ldots$
193	11	Read: For an assigned value...
193	13	$\widehat{\mu}_r = \int_0^\infty x^r y\, dx.$
193	25	Read: $\mu_2 > \lambda(M_{2,n})^{1/n}$
193	28	Read: inequality
194	10	Change parenthesis to read: $(\ldots + \frac{r^2}{2}\beta_{2,4})$
194	16	Last term of exponent should read: $-\dfrac{2rxy}{\sigma_x\sigma_y}$.
195	14	Change us to up.

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

PROPOSALS

243. *Proposed by William R. Ransom, Tufts College.*

In X:2 Euclid proves that the secant of 45° cannot be expressed as the ratio of integers. Using this method of proof, bisecting the angle and infinite descent, show that the tangent of 30° cannot be expressed as the ratio of integers.

244. *Proposed by P.A.Piza, San Juan, Puerto Rico.*

Take any nine consecutive positive integers and find among them (with only three duplications) two sets of six integers such that their sums, the sums of the squares and the sums of their cubes are equal.

245. *Proposed by Chih-yi Wang, University of Minnesota.*

Evaluate $\int_0^{\pi/4} \sqrt{\tan x}\, dx$ without using approximation methods.

246. *Proposed by A.S.Gregory, University of Illinois.*

Let a function be defined by $b\,f(x) - ax = \sin\,(a\,f(x) + bx)$ with $0 < a < b$, $a^2 + b^2 = 1$. Expand $f(x)$ in a Maclaurin's Series.

247. *Proposed by Julian H. Braun, White Sands Proving Ground, New Mexico.*

Reduce $f(n) = \sum_{k=1}^{[3n/4]} \sum_{i=1}^{u} 3i$ to the form $[g(n)]$ where $u = n - [(4k-1)/3]$, $g(n)$ is a polynomial and $[x]$ denotes the greatest integer less than or equal to x.

248. *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Let Γ_1 and Γ_2 be two plane curves. Let t be a variable line intersecting these curves at the points M_1, M_2 where the tangents t_1 and t_2 to the curves are parallel to each other. Prove that the centers of curvature C_1 and C_2 of Γ_1 and Γ_2 at M_1 and M_2 are collinear with the characteristic point C of the straight line t.

249. *Proposed by David Sayre, IBM Corp, New York.*

The zeros of a polynomial $P(z)$, of degree n^2, all lie on the unit circle and are expressable as $\exp i(a_j - a_k)$ with $j,k = 1,2,3,\ldots n$. The a's are a set of n unknown real numbers. What can be said about the coefficients of the polynomial?

SOLUTIONS

Late Solutions

217. *Sister M. Stephanie, Georgian Court College, New Jersey.*

A Non-integral Triangle

223. [January 1955] *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn.*

Prove that there is no integral triangle such that $\cos A \cos B + \sin A \sin B \sin C = 1$.

Solution by Chih-yi Wang, University of Minnesota. The relation $(\cos A - \cos B)^2 = (1 - \cos^2 A)(1 - \cos^2 B)(1 - \cos^2 C)$ implies that (1) $(\cos A - \cos B)^2 = -\cos^2 C \sin^2 A \sin^2 B$.

The equation (1) has the trivial solution $A = B = 0°$, $C = 180°$, and the only non-trivial solution $A = B = 45°$, $C = 90°$. Since the two legs and hypotenuse of an isosceles right triangle are proportional to 1,1, $\sqrt{2}$ the stated result follows.

Also solved by S. H. Sesskin, Hofstra College; E. P. Starke, Rutgers University. and the proposer.

Newton's Approximation

224. [January 1955] *Proposed by Ben B. Bowen, Vallejo College, Calif.*

Find the radius of a circle when a chord, whose maximum distance from the circumference is ten feet, cuts off an arc of 160 feet.

Solution by S. H. Sesskin Hofstra College. Let arc and chord be AB, and the central angle they subtend be θ. Then the perpendicular bisector of AB passes through the center of the required circle, the radius of which is $x + 10$. Now since $S = r\theta$ we have $160 = \theta(x + 10)$.

But $\theta = 2 \text{ arc cos } \dfrac{x}{x+10}$ so $\dfrac{80}{x+10} = \text{arc cos } \dfrac{x}{x+10}$.

By Newton's method we find a root at approximately $x = 308$ feet. Hence the radius is 318 feet.

Also solved by C. E. Jones, Tennessee A & I State University; M. S. Klamkin, Brooklyn Polytechnic Institute; Sam Kravitz, East Cleveland, Ohio: L. A. Ringenberg, Eastern Illinois State College and the Proposer.

Concentric Ellipses

226. [January 1955] *Proposed by P. D. Thomas, Eglin Air Force Base, Florida.*

Tangents are drawn from a point P to an ellipse. If R and Q are the points of contact and O is the center of the ellipse, find the locus of P if the area of the the quadrilateral $PQOR$ remains constant.

I. *Solution by Dennis C. Russell, Birkbeck College, University of London.* In parametric form with the eccentric angle ϕ as parameter, let $Q = (a \cos \phi_1, b \sin \phi_1)$, $R = (a \cos \phi_2, b \sin \phi_2)$, $P = (X, Y)$. Then the area of the triangle OPQ is the numerical value of

$$\Delta_1 = \tfrac{1}{2} \begin{vmatrix} X & Y \\ a \cos \phi_1 & b \sin \phi_1 \end{vmatrix} = \tfrac{1}{2}(bX \sin \phi_1 - aY \cos \phi_1)$$

and the area of the triangle OPR is the numerical value of

$$\Delta_2 = \tfrac{1}{2}(bX \sin \phi_2 - aY \cos \phi_2).$$

Since Q, R are on opposite sides of the line OP, Δ_1 and Δ_2 will be opposite in sign and so the area A of the quadrilateral $PQOR$ is the numerical value of $\Delta_1 - \Delta_2$; that is

$$\begin{aligned} \pm A &= \tfrac{1}{2}bX(\sin \phi_1 - \sin \phi_2) + \tfrac{1}{2}aY(\cos \phi_1 - \cos \phi_2) \\ &= [bX \cos \tfrac{1}{2}(\phi_1 + \phi_2) + aY \sin \tfrac{1}{2}(\phi_1 + \phi_2)] \sin \tfrac{1}{2}(\phi_1 - \phi_2). \end{aligned}$$

For brevity, write

$$\tfrac{1}{2}(\phi_1 + \phi_2) = \psi, \quad \tfrac{1}{2}(\phi_1 - \phi_2) = \alpha$$

then $(bX \cos \psi + aY \sin \psi) \sin \alpha = \pm A = \text{constant}$.......$(i)$

The equations of the tangents at Q and R are, by the well-known formulae,

$$bX\cos\phi_1 + aY\sin\phi_1 = ab$$
$$bX\cos\phi_2 + aY\sin\phi_2 = ab$$

and it is a simple matter to solve these for X and Y to give

$$X = a\cos\psi\sec\alpha, Y = b\sin\psi\sec\alpha \quad \ldots\ldots\ldots\ldots\ (ii)$$

Substituting these values of X and Y in (i),

$$(ab\cos^2\psi + ab\sin^2\psi)\tan\alpha = \text{constant}$$

that is $\tan\alpha$ = constant and so α = constant.

Hence equation (ii) gives the parametric equations of the locus of P, which is therefore the concentric ellipse

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = \sec^2\alpha = 1 + \tan^2\alpha = 1 + \frac{A^2}{a^2b^2}$$

that is, the required locus is the concentric ellipse

$$b^2X^2 + a^2Y^2 = A^2 + a^2b^2,$$

where A is the area of the quadrilateral $PQOR$.

II. *Solution by Huseyin, Demir, Zonguldak, Turkey.* The ellipse is an orthogonal projection of a circle. Let $P'Q'OR'$ be the corresponding quadrangle. The locus of P' is a concentric circle, for the two quadrangles are in a constant ratio (in area). Hence the locus of P, projection of P', is an ellipse homothetic with the original one.

Also solved by M. S. Klamkin, Brooklyn Polytechnic Institute; S. H. Sesskin, Hofstra College; E. P. Starke, Rutgers University; Chih-yi Wang, University of Minnesota and the proposer.

Forces In Equilibrium

227. [January 1955] *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Let A_1B_1, A_2B_2 and A_3B_3 be three bars of lengths l_1, l_2 and l_3 with weights W_1, W_2 and W_3 respectively. The ends B_1, B_2 and B_3 rest on a horizontal surface while the other ends A_1, A_2 and A_3 are supported by the bars A_3B_3, A_1B_1 and A_2B_2 respectively. Find the reactions R_1, R_2 and R_3 at B_1, B_2 and B_3.

Solution by the proposer. Let the reactions of the bars at the ends A_1, A_2, A_3 be denoted by r_1, r_2, r_3 and the lengths A_1A_2, A_2B_1: A_2A_3, A_3B_2; A_3A_1, A_1B_3 by a_1, b_1; a_2, b_2; a_3, b_3 respectively.

Then considering the equilibrium of one of the bars, say A_1B_1, we have by taking moments of the forces r_1, r_2, W_1, R_1 at the point B_1:

$$\tfrac{1}{2}\, l_1W_1 - l_1r_1 + b_1r_2 = 0.$$

Setting $b_i = k_i l_i$ $(i = 1,2,3)$ and considering the other equations corresponding to the two other bars, we get the system of equations with unknowns r_1, r_2, r_3:

$$r_1 - k_1r_2 = \tfrac{1}{2}W_1$$

$$r_2 - k_2r_3 = \tfrac{1}{2}W_2$$

$$r_3 - k_3r_1 = \tfrac{1}{2}W_3$$

The determinant of this system being

$$D = \begin{vmatrix} 1 & -k_1 & 0 \\ 0 & 1 & -k_2 \\ -k_3 & 0 & 1 \end{vmatrix} = 1 - k_1k_2k_3$$

we have

$$r_1 = (W_1 + k_1W_2 + k_1k_2W_3)/2D$$

$$r_2 = (W_2 + k_2W_3 + k_2k_3W_1)/2D$$

$$r_3 = (W_3 ++ k_3W_1 + k_3k_1W_2)/2D$$

and

$$R_1 = W_1 - r_1 + r_2 = W_1 - (W_1 + k_1W_2 + k_1k_2W_3)/2D + W_2 + k_2W_3 + k_2k_3W_1)/2D$$

$$R_1 = [(1 + k_2k_3 - 2k_1k_2k_3)W_1 + (k_1 - 1)W_2 + k_2\, k_1 - 1)W_3]/2(1 - k_1k_2k_3)$$

$$R_2 = [k_3(k_2 - 1)W_1 + (1 + k_3k_1 - 2k_1k_2k_3)W_2 + (k_2 - 1)W_3]/2(1 - k_1k_2k_3)$$

$$R_3 = [(k_3 - 1)W_1 + k_1(k_3 - 1)W_2 + (1 + k_1k_2 - 2k_1k_2k_3)W_3]/2(1 - k_1k_2k_3)$$

Also solved by George R. Mott, Republic Aviation Company.

An Arc Tangent Function

206. [May 1954] *Proposed by W. E. Byrne, Lexington, Virginia.*

If the symbol 'arc tan u' is defined to be the angle (in radians) between $-\pi/2$ and $\pi/2$ whose tangent is u, determine for the interval $-\pi \le x \le \pi$ the various values of the constant in the formula

$$2\sigma = \alpha + \beta + \text{constant}$$

where

$$\alpha = \text{arc tan } (\tfrac{1}{2}\sqrt{3} \tan x), \quad \beta = \text{arc tan } (\cos x/\sqrt{3})$$

$$\sigma = \text{arc tan } \frac{2 \tan \tfrac{1}{2} + 1}{\sqrt{3}}.$$

Solution by H. E. Fettis, Dayton, Ohio. Let the constant in the formula be designated by $-2C$. Then $2(\sigma - C) = \alpha + \beta$.

Now

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{\sqrt{3}/2(\tan x) + \sqrt{3}/3(\cos x)}{1 - \tfrac{1}{2}\sin x}$$

$$= \frac{\sqrt{3}}{3}\left[\frac{1 + 2\sin x}{\cos x}\right]$$

whence

$$\sec(\alpha + \beta) = \sqrt{1 + \frac{1}{3}\left(\frac{1 + 4\sin x + 4\sin^2 x}{\cos^2 x}\right)}$$

$$= \pm\frac{2 + \sin x}{\sqrt{3}\cos x}$$

Taking first the positive sign, we then have

$$\tan\left(\frac{\pi}{4} + \frac{\alpha + \beta}{2}\right) = \sec(\alpha + \beta) + \tan(\alpha + \beta)$$

$$= \sqrt{3}\,\frac{1 + \sin x}{\cos x}$$

$$= \sqrt{3}\,\frac{1 + \tan\tfrac{1}{2}x}{1 - \tan\tfrac{1}{2}x}$$

and since

$$\tan\sigma = \frac{2\tan\tfrac{1}{2}x + 1}{\sqrt{3}}$$

or

$$\tan\tfrac{1}{2}x = \tfrac{1}{2}(\sqrt{3}\tan\sigma - 1)$$

we have

$$\tan\left(\frac{\pi}{4}+\frac{\alpha+\beta}{2}\right)=\sqrt{3}\,\frac{1+\tfrac{1}{2}(\sqrt{3}\tan\sigma-1)}{1-\tfrac{1}{2}(\sqrt{3}\tan\sigma-1)}$$

$$=\frac{\sqrt{3}/3+\tan\sigma}{1-\sqrt{3}/3\tan\sigma}$$

$$=\tan(\pi/6+\sigma)$$

Therefore either

$$\frac{\pi}{4}+\frac{\alpha+\beta}{2}=\frac{\pi}{6}+\sigma$$

whence

$$C=\frac{\pi}{12}$$

or

$$\pi+\frac{\pi}{4}+\frac{\alpha+\beta}{2}=\frac{\pi}{6}+\sigma$$

whence

$$C=\frac{13\pi}{12}$$

Similarly, taking the negative sign before the radical,

$$\tan\left(\frac{\pi}{4}+\frac{\alpha+\beta}{2}\right)=\frac{\tan\sigma-\sqrt{3}}{1+\tan\sigma\sqrt{3}}$$

$$=\tan(\sigma-\pi/3)$$

giving either

$$C=7\frac{\pi}{12}$$

or

$$C=-5\frac{\pi}{12}$$

To identify the various solutions with the various values of x, we first note that the absolute value of $(\alpha+\beta)$ can never exceed $\pi/2$, and therefore that $C=\sigma-(\alpha+\beta)/2$ can never exceed π. The solution $C=(13\pi)/12$ is therefore ruled out by the initial conditions of the problem. Also, for values of x between 0 and $\pi/12$, $(\alpha+\beta)$ lies between $\pi/6$ and $\pi/2$, while for values of x between 0 and $-\pi/2$, $(\alpha+\beta)$ lies between $\pi/6$ and $-\pi/2$. Since sec $(\alpha+\beta)$ and cos x are both positive in these intervals, the choice of the positive sign before the radical gives the value of C in these two intervals. And since the value $C=13(\pi/12)$ has been eliminated, it follows that for

$$0 \leq x \leq \pi/2, \; C = \pi/12$$

$$0 > x \geq -\pi/2, \; C = \pi/12$$

For the intervals $\pi/2 < x \leq \pi$, and $-\pi/2 > x \geq -\pi$, sec $(\alpha + \beta)$ is positive and cos x is negative, and it is clear that, since σ is positive only in the former of these, for

$$\pi/2 < x \leq \pi, \; C = 7(\pi/12)$$

$$-\pi/2 > x \geq -\pi, \; C = -5(\pi/12).$$

Also solved by the proposer.

A Product of Two Binomials

218. [November 1954] *Proposed by Ben K.Gold, Los Angeles City College.*

Prove $$\sum_{i=0}^{K} (-1)^i \binom{K+i}{K}\binom{2K+1}{K-i} = 1$$

Additional solution by T.F.Mulcrone, St. Charles College, Louisiana.

The method employed makes use of generalized figurate numbers, F_i^n, of order n. See *American Mathematical Monthly*, Vol. 57 (1950), page 14-23.

Thus we have: $$(-1)^i \binom{2K+1}{K-i} = (-1)^K F_{K-i}^{-(2K+1)}$$

$$\binom{K+i}{K} = \binom{K+i}{i} = F_i^{K+1}$$

And hence the given summation becomes:

$$(-1)^K \sum_{i=0}^{K} F_{K-i}^{-(2K+1)} \cdot F_i^{K+1}$$

$$= (-1)^K F_K^{-(2K+1)} \cdot F_K^{K+1}$$

$$= (-1)^K F_K^{-K}$$

$$= \binom{K}{K}$$

$$= 1.$$

Other solutions appear in Vol. 28, No. 5, May 1955, page 288 of this magazine.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and source, if known.

Q 146. Sum the series $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots$ [*Submitted by M.S.Klamkin*]

Q 147. If it is known that a is positive, what can be said about the value of the periodic continued fraction $(5;1,a,1,10,1,\dots)$? [*Submitted by Norman Anning*].

Q 148. Express $1/(1+x)(1+x^2)(1+x^4)(1+x^6)$ as a power series. [*Submitted by M. S. Klamkin*].

Q 149. Sum $S(x) = 1/x + (1+1/x) + (x+1+1/x) + \dots + (x^{n-2} + x^{n-3} + \dots + 1 + 1/x)$. [*Submitted by Barney Bissinger*].

Q 150. Show that $(a+b+c)^3 = 27\,abc$ if $a^{1/3}+b^{1/3}+c^{1/3}=0$, [*Submitted by M.S.Klamkin*].

Q 151. Taking recripocals of integers the following series was discovered: .0123456790123... . What is the number? [*Submitted by Glenn D. James*].

ANSWERS

A 149. $$S(x) = \frac{1}{x}\left[\frac{x-1}{x-1}+\frac{x^2-1}{x-1}+\dots+\frac{x^n-1}{x-1}\right]$$

A 148. $$1/(1+x)(1+x^2)(1+x^4)(1+x^6) = (1-x)/(1-x^{16})$$
$$= (1-x)(1+x^{16}+x^{32}+x^{48}+\dots)$$
$$= 1 - x + x^{16} - x^{17} + x^{32} - x^{33} + \dots$$

A 147. The value is $\sqrt{\frac{36a+60}{a+2}}$ and consequently is confined between $\sqrt{30}$ and $\sqrt{36}$.

A 146. $$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots = 1.$$

A 151. From the location of the decimal point the number must contain two digits. Since 123 has three significant figures, take its reciprocal to get 81 which is the answer.

A 150. $a + b + c - 3(abc)^{1/3} = (a^{1/3} + b^{1/3} + c^{1/3})(a^{2/3} + b^{2/3}$ $c^{2/3} - a^{1/3}b^{1/3} - b^{1/3}c^{1/3} - c^{1/3}a^{1/3})$. Now if $a^{1/3} + b^{1/3} + c^{1/3}$ $= 0$ then we have $a + b + c = 3(abc)^{1/3}$ or $(a + b + c)^3 = 27abc$.

$$= \frac{x^{n-1}}{(x-1)^2} - \frac{n}{x(x-1)} \quad \text{for } x \neq 0,\ 1 \text{ and } S(1) = n(n+1)/2.$$

TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase or idea rather than upon a mathematical routine. Send us your favorite trickies.

T 19. A thermometer registers zero. What would it register if it were twice as cold? [*Submitted by T. F. Mulcrone*].

T 20. A trip is made at a speed of 30 mph. At what speed must the return trip be made in order to average 60 mph? [*Submitted by J. M. Howell*].

SOLUTIONS

S 20. If the distance one way is x miles, then the time going is $x/30$ hours and the over-all time is $2x/60$ hours. Since these times are equal there is no time available for the return journey. The situation is impossible.

S 19. It is from the freezing point of water, 32 degrees Fahrenheit, that we measure heat and cold. Hence at zero it is 32 degrees cold, and twice as cold would be 32 degrees below zero.

QUICKENING THE QUICKIES

Q 142. Find the class of functions such that $\frac{1}{F(x)} = F(-x)$

[*Alternate solution by Gaines Lang.*] By taking absolute values and then logarithms of each side it is clear that $\ln F(x)$ is an odd function. Hence $F(x) = \pm e^{G(x)}$ where $G(x)$ is an odd function.

(Current Papers - Continued from page 40)

COMMENT ON BIRNBAUM AND OMMIDVAR'S "THE GROUP METHOD"*

H. W. Becker

This method is standard equipment in electronics lab courses, as taught by the writer at Mare I. Navy Yard and Omaha, and prob'ly the world over. The optimum group size is 3, the actual number depending on the ratio of manpower to gear. It is necessary that the groups be of the same average ability and speed, so all keep pace with the daily schedule of theory briefings on the experiments. The men group themselves, usually a cluster of "greenhorns" around each "sharpy". But sometimes a couple of experienced "hams" will team up, and tend to outstrip the field; it suffices to suggest they break it up and choose some other partner, to stay in phase with the class.

Strangely enough, it never occurred to the writer to adapt this method to math. instruction. Birnbaum and Ommidvar deserve medals for their innovation and their write-up of its merits could hardly be improved.

* Mathematics Magazine, 28 (1955) 277-9.

CALCULATING MACHINE - ABACUS

A. M. Maish

Managing Editor
Mathematics Magazine

Dear Sir:

In the November-December 1954 issue, (Volume 28, No. 2) on page 83, there is an article by Cpl. Jerry Adler that contains some errors of fact. I was present at the contest in November 1946 in the Ernie Pyle theater. In fact, I supplied the calculating machine, a model S-10 Friden, that had been through the war and had been reconditioned. I did not, however supply the operator. Due to the rapid turnover of personnel I had nobody available with as much as one month's practice on the machine, so I yielded to another organization.

The Finance Office supplied a corporal who was careful and slow. He had been taught to check and double check. During the contest, he would read one digit, turn his head to the machine, press the correct key, and so on. Sometimes he was still confirming his input when "Hands" Matzuzaki was scribbling the answer with one hand and shaking his abacus with the other to clear the rods for the next problem.

I think anyone in the audience, children included, could have bettered the corporal's performance if given one hour's instruction. The difference in the final result was largely due to difference in ability

(Current Papers - Continued from page 40)

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Alexander M. Maish
Major, C.E.
Instructor
Department of Mathematics

Stochastic Models for Learning. By Robert R. Bush and Frederick Mosteller, John Wiley & Sons, New York, 365 pp., $9.00.

The latest addition to the Wiley Publications in Statistics (Walter A. Shewhaft and S.S. Wilks, editors), this new book offers a possible probabilistic framework or model for analyzing data from a variety of experiments on animal and human learning.

Bush and Mosteller stress the fact that learning is probabilistic or stochastic, with some events increasing and others decreasing the probability of certain responses. In the first part of their book, the authors present the general model, describe many of its formal properties, and consider a number of special cases. The individual chapters in this section are devoted to the basic model, stimulus sampling and conditioning, sequences of events, distributions of response probabilities, the equal alpha condition, approximate methods, operators with limits zero and unity, and commuting operators.

Part II applies the model to a number of specific experiments, and treats the statistical problems of estimating parameters and measuring goodness of fit. In this half of the book, the chapters cover identification and estimation, free-recall verbal learning, avoidance training, an experiment of imitation, symmetric choice problems, runway experiments, and evaluations.

Dr. Bush, assistant professor of social relations at Harvard University, was formerly at Princeton where he helped construct the Princeton cyclotron. In addition to teaching, he has been associated with the RCA Laboratories and is the author of numerous articles in major magazines.

Dr. Mosteller, professor of mathematical statistics at Harvard, was earlier in the Office of Public Opinion Research and a consultant to the Research Branch of the War Department. Often affiliated with leading journals in the field, he has also written many papers on the theory and applications of statistics to psychology, industrial quality control, and biostatistics.

Richard Cook

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Part II applies the model to a number of specific experiments, and treats the statistical problems of estimating parameters and measuring goodness of fit. In this half of the book, the chapters cover identification and estimation, free-recall verbal learning, avoidance training, an experiment of imitation, symmetric choice problems, runway experiments, and evaluations.

Dr. Bush, assistant professor of social relations at Harvard University, was formerly at Princeton where he helped construct the Princeton cyclotron. In addition to teaching, he has been associated with the RCA Laboratories and is the author of numerous articles in major magazines.

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